Chapter 2

Measuring Liquidity
2.8 Exercises

2. Measures of the bid-ask spread. Your fund is considering trading 10-year bonds issued by the Austrian government, and you see that at 9:30 a.m. their lowest ask price is 102.31 and their highest bid price is 99.50. Five seconds later a buy order for a block of €10 billion is executed at 102.76. At 10:30 a.m. you check the market again and see that the lowest ask price is 102.55 and the highest bid price is 100.02.

   a. Compute the absolute and the relative quoted spread at 9:30 and 10:30.

   b. Compute the absolute and the relative effective ask-side half-spread at 9:30.

   c. Compare the quoted half-spread with the effective ask-side half-spread (both in absolute and in relative terms) at 9:30. What explains the difference between them?

   d. Compute the absolute realized spread in the 9:30–10:30 interval.

   e. Compare the realized spread computed under point d with the absolute effective spread at 9:30 computed under point b. What explains the difference between them?

4. Roll’s estimator and price improvements. Consider Roll’s model presented in section 2.3.4. All the assumptions are unchanged, but we do assume that the transaction occurring at time $t$ occurs either at the ask or bid price with probability $\lambda$, or at the midprice with probability $1 - \lambda$. Thus, $1 - \lambda$ can be seen as the fraction of trades that receive a price improvement or are crossed by brokers at the midprice. Propose an estimator of the quoted bid-ask spread $S$ for this case.
2.9 Solutions

Exercise 2:

a. At 9:30 the absolute spread is 2.81, and the relative spread (that is equal to the absolute one divided by the midquote) is 2.78%. At 10:30 the absolute spread is 2.53 and the relative spread is 2.5% (see calculation details in file Ch2_ex2_solutions.xls).

b. At 9:30 the absolute effective ask-side half-spread is 1.85, and the relative effective ask-side half-spread is 1.84% (see calculation details in file Ch2_ex2_solutions.xls).

c. At 9:30 the relative quoted half-spread is 2.78%/2 = 1.39%, while the relative effective spread is 1.84%. The difference may be explained by the fact that the buy order associated with the transaction price of 102.76 is very large, and therefore “eats” up several limit orders on the buy side of the LOB (if the market is organized as an order-driven market) or equivalently induces dealers to require a higher ask price than the quote one to supply the desired amount. In other words, the difference between the two spreads tells us that, on the ask side, the market for these bonds has limited depth.

d. The realized ask-side half-spread is 1.475 (see calculation details in file Ch2_ex2_solutions.xls).

e. The realized half-spread is lower than the effective one. The difference is due to the fact that following the large buy order at 9:30:05, both the bid and the ask prices have increased, possibly reflecting the informational content of the order. As usual in these cases, the realized spread (which measures the profits of the market maker, benchmarked against a later price) is lower than the effective spread (which measures the trading cost to a trader placing a market order). The reason is that after the order execution, the market price “turns against” the dealer who has filled the buy order.
Exercise 4: Assuming that the midprice follows a random walk, that is $m_t = m_{t-1} + \epsilon_t$, with $E[\Delta m_t] = 0$ and $E[\epsilon_t, \epsilon_{t-1}] = 0$, the transaction price will be:

$$p_t = \begin{cases} 
  a_t = m_t + S/2 & \text{if } d_t = 1, \text{ that happens with probability } \frac{\lambda}{2}, \\
  m_t & \text{if } d_t = 0, \text{ that happens with probability } \frac{1-\lambda}{2}, \\
  b_t = m_t - S/2 & \text{if } d_t = -1, \text{ that happens with probability } \frac{\lambda}{2}.
\end{cases}$$

Therefore,

$$p_t = m_t + \frac{S}{2} d_t$$

and

$$\Delta p_t = m_t + \frac{S}{2} \Delta d_t = \frac{S}{2} \Delta d_t + \epsilon_t$$

Notice that Roll’s model assumptions are satisfied since:

- the order flow is balanced:
  
  $$\Pr(d_t = 1) = \Pr(d_t = -1);$$

- there is no autocorrelation in orders:
  
  $$E[d_t, d_s] = 0 \text{ for } t \neq s;$$

- market orders have no effect on the midquote:
  
  $$E[d_t, \epsilon_t] = E[d_t, \epsilon_{t+1}] = 0;$$

- the expected return is constant and equal to 0:
  
  $$E[\Delta m_t] = 0 \Rightarrow E[\Delta p_t] = 0.$$
If follows that
\[
\text{cov}(\Delta p_{t+1}, \Delta p_t) = E\left[\left(\frac{S}{2} \Delta d_{t+1} + \epsilon_{t+1}\right)\left(\frac{S}{2} \Delta d_t + \epsilon_t\right)\right] - E(\Delta p_{t+1})E(\Delta p_t) =
\]
\[
= \frac{S^2}{4} E(\Delta d_{t+1} \Delta d_t) + \frac{S}{2} E(\Delta d_{t+1} \epsilon_t) + \frac{S}{2} E(\Delta d_t \epsilon_{t+1}) + E(\epsilon_{t+1} \epsilon_t) =
\]
\[
= \frac{S^2}{4} E((d_{t+1} - d_t)(d_t - d_{t-1})) =
\]
\[
= -\frac{S^2}{4} E(d_t^2)
\]
and, given the probability distribution of \(d_t\):
\[
E(d_t^2) = \frac{\lambda}{2} 1^2 + \frac{\lambda}{2} (-1)^2 + \frac{1 - \lambda}{2} 0^2 = \lambda
\]
Therefore,
\[
\text{cov}(\Delta p_{t+1}, \Delta p_t) = -\frac{S^2}{4} \lambda
\]
and the spread is
\[
S = 2 \sqrt{\frac{-\text{cov}(\Delta p_{t+1}, \Delta p_t)}{\lambda}},
\]
which exceeds Roll’s measure of the spread, i.e. \(S_R = 2 \sqrt{-\text{cov}(\Delta p_{t+1}, \Delta p_t)}\): the true spread is \(S = S_R/\sqrt{\lambda} > S_R\), as \(\lambda < 1\). In other words, Roll’s measure underestimates the spread paid by those who cannot obtain the price improvement, because it is based also on data for those who can do so (and in fact pay a zero spread).