

6

You Cannot be Series

The limiting process was victorious. For the limit is an indispensable concept, whose importance is not affected by the acceptance or rejection of the infinitely small.

Philosophy of Mathematics and Natural Science, Hermann Weyl

6.1 What are Series?

Sequences are the fundamental objects in the study of limits. In this chapter we will meet a very special type of sequence whose limit (when it exists) is the best meaning we can give to the intuitive idea of an ‘infinite sum of numbers’. Let’s be specific. Suppose we are given a sequence (a_n) . Our primary focus in this chapter will not be on the sequence (a_n) but on a related sequence which we will call (s_n) . It is defined as follows:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3,$$

and more generally

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

The sequence (s_n) is called a *series* (or infinite series) and the term s_n is often called the *n*th partial sum of the series. The goal of this chapter will be to investigate when the series (s_n) has a limit. In this way we can try to give meaning to ‘infinite sums’ which we might write informally as

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots, \tag{6.1.1}$$

$$1 - 1 + 1 - 1 + 1 - 1 + \dots, \quad (6.1.2)$$

but beware that the meaning that we'll eventually give to expressions like this (in those cases where it is indeed possible) will not be the literal one of an infinite number of additions (what can that mean?), but as the limit of a sequence. Sequences that arise in this way turn up throughout mathematics and its applications so understanding them is very important.

6.2 The Sigma Notation

Before we can start taking limits of series we need to develop some useful notation for finite sums that will simplify our approach to expressions like (6.1.1) and (6.1.2).¹

Let's suppose that we are given the following ten whole numbers: $a_1 = 3$, $a_2 = -7$, $a_3 = 5$, $a_4 = 16$, $a_5 = -1$, $a_6 = 3$, $a_7 = 10$, $a_8 = 14$, $a_9 = -2$, $a_{10} = 0$. We can easily calculate their sum

$$a_1 + a_2 + \dots + a_{10} = 41, \quad (6.2.3)$$

but there is a more compact way of writing the left-hand side which is widely used by mathematicians and those who apply mathematics. It is called the 'sigma notation' because it utilises the Greek letter Σ (pronounced 'sigma') which is capital S in English (and S should be thought of here as standing for 'sum'). Using this notation we write the left-hand side of (6.2.3) as

$$\sum_{i=1}^{10} a_i.$$

Now if you haven't seen it before, this may appear to be a complex piece of symbolism – but don't despair. We'll unpick it slowly and we'll read bottom up. The $i = 1$ tells us that the first term in our addition is a_1 , then the Σ comes into play and tells us that we must add a_2 to a_1 and then a_3 to $a_1 + a_2$ and then a_4 to $a_1 + a_2 + a_3$ and then ... but when do we stop? Well go to the top of Σ and read the number 10. That tells you that a_{10} is the last number you should add, and that's it. The notation is very flexible. For example you also have

$$\sum_{i=1}^8 a_i = a_1 + a_2 + \dots + a_8 = 43,$$

$$\sum_{i=2}^9 a_i = a_2 + a_3 + \dots + a_9 = 38,$$

¹ Readers who already know about this notation may want to omit this section.

$$\sum_{i=6}^7 a_i = a_6 + a_7 = 13.$$

The following results are easily derived and quite useful. We will use them without further comment in the sequel. If $a_1, \dots, a_n, b_1, \dots, b_n$ and c are arbitrary real numbers then

$$\begin{aligned}\sum_{i=1}^n (a_i + b_i) &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i, \\ \sum_{i=1}^n ca_i &= c \sum_{i=1}^n a_i.\end{aligned}$$

The sigma notation is particularly useful when we have a sequence (a_n) and we want to add as far as the general term a_n to obtain the n th partial sum s_n . We can now write

$$s_n = \sum_{i=1}^n a_i.$$

Notice that the letter i here is only playing the role of a marker in telling us where to start and stop adding. It is called a 'dummy index' and the value of s_n is unchanged if it is replaced by a different letter throughout – e.g.

$$\sum_{i=1}^n a_i = \sum_{j=1}^n a_j.$$

It's worth pointing out one special case that often confuses students. Suppose that $a_i = k$ takes the same value for all i . Then

$$\sum_{i=1}^n a_i = \sum_{i=1}^n k = \underbrace{k + k + \dots + k}_{n \text{ times}} = nk.$$

Our main concern in this chapter is with infinite rather than finite series, but before we return to the main topic let's look at an interesting problem that (and this may be an apocryphal story) was given to one of the greatest mathematicians the world has even seen, Carl Friedrich Gauss (1777–1855),² when he was a schoolboy. The story goes that the teacher wanted to concentrate on some urgent task and so he asked the whole class to calculate the sum of the first 100 natural numbers so that he could work in peace while they struggled with this fiendish problem. Apparently after 5 minutes Gauss produced the correct answer 5050. How did he get this? It is speculated that he noticed the following clever pattern

² See e.g. <http://202.38.126.65/navigate/math/history/Mathematicians/Gauss.html>

by writing the numbers 1 to 50 in a row and then the numbers 51 to 100 in reverse order underneath:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & 49 & 50 \\ 100 & 99 & 98 & 87 & \cdots & 52 & 51 \end{array}$$

Now each column adds to 101 – but there are 50 columns and so the answer is $50 \times 101 = 5050$. In sigma notation, Gauss calculated $\sum_{i=1}^{100} i$. A natural generalisation

is to seek a formula for $\sum_{i=1}^n i$, i.e. the sum of the first n natural numbers. If $n = 2m$ is an even number you can use exactly the same technique as Gauss to show that the answer is $m(2m + 1)$ or $\frac{1}{2}n(n + 1)$. The same is true when $n = 2m + 1$ is odd since (using the fact that we know the answer when n is even),

$$\begin{aligned} \sum_{i=1}^{2m+1} i &= \sum_{i=1}^{2m} i + (2m + 1) \\ &= m(2m + 1) + (2m + 1) \\ &= (m + 1)(2m + 1) \\ &= \frac{1}{2}n(n + 1). \end{aligned}$$

So we've shown that for every natural number n

$$\sum_{i=1}^n i = \frac{1}{2}n(n + 1). \quad (6.2.4)$$

At this stage it's worth briefly considering more general finite sums which are obtained from summing all the numbers on the list $a, a + d, a + 2d, \dots, a + (n - 1)d$. Note that there are n numbers in this list which are obtained from the *first term* a by repeatedly adding the *common difference* d . Such a list is called an *arithmetic progression* and we can find the sum S of the first n terms by using a slight variation of Gauss' technique. So we write

$$\begin{array}{l} S = \quad a \quad + \quad (a + d) \quad + \quad (a + 2d) \quad + \cdots + [a + (n - 1)d] \\ S = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \cdots + \quad a \end{array}$$

Now add these two expressions, noting that each of the n columns on the right-hand side sums to $2a + (n - 1)d$ to get

$$2S = n(2a + (n - 1)d),$$

and so

$$S = n \left[a + \frac{1}{2}(n - 1)d \right].$$

If you take $a = d = 1$, you can check that you get another proof of (6.2.4).

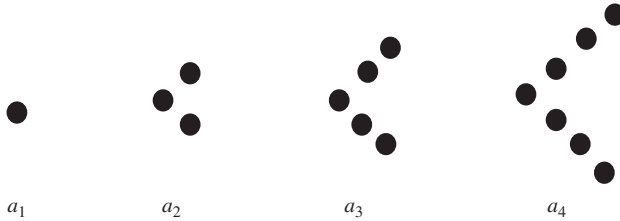


Figure 6.1. Representation of some triangular numbers.

Before we return to our main topic, I can't resist introducing the *triangular numbers*. This is the sequence (a_n) defined by $a_1 = 1$, $a_2 = 1 + 2 = 3$, $a_3 = 1 + 2 + 3 = 6$, $a_4 = 1 + 2 + 3 + 4 = 10$, etc. so the n th term is $a_n = \frac{1}{2}n(n+1)$. Figure 6.1 above should help you see why these numbers are called 'triangular'.³

It is a fact that if you add successive triangular numbers you always get a perfect square so e.g.

$$1 + 3 = 4 = 2^2$$

$$3 + 6 = 9 = 3^2$$

$$6 + 10 = 16 = 4^2.$$

It is easy to prove that this holds in general by using (6.2.4). Indeed we have that the sum of the n th and $(n+1)$ th triangular numbers is

$$\begin{aligned} a_n + a_{n+1} &= \frac{1}{2}n(n+1) + \frac{1}{2}(n+1)(n+2) \\ &= \frac{1}{2}(n+1)(n+n+2) \\ &= (n+1)^2. \end{aligned}$$

6.3 Convergence of Series

Let's return to the main topic of this chapter. We are given a sequence (a_n) and we form the associated sequence of partial sums (s_n) where $s_n = \sum_{i=1}^n a_i$. Now suppose that (s_n) converges to some real number l in the sense of Chapter 4, i.e. for any $\epsilon > 0$ there exists a natural number n_0 such that whenever $n > n_0$ we have $|s_n - l| < \epsilon$. In this case we call l the *sum of the series*. In some ways this is a bad name as l is not a sum in the usual sense of the word, it is a limit of sums,

³ See also http://en.wikipedia.org/wiki/Triangular_number.

but this is standard terminology and we will have to live with it. There is also a special notation for l . We write it as $\sum_{i=1}^{\infty} a_i$. Again from one point of view, this is a bad notation as it makes it look like an infinite process of addition, but on the other hand it is pretty natural once you get used to it and it works well from the following perspective:

$$l = \sum_{i=1}^{\infty} a_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i = \lim_{n \rightarrow \infty} s_n.^4$$

Just to be absolutely clear that we understand what $\sum_{i=1}^{\infty} a_i$ is I'll remind you that (if it exists) it is the real number that has the property that given any $\epsilon > 0$ there exists a natural number n_0 such that whenever $n > n_0$ we have $\left| \sum_{i=1}^n a_i - \sum_{i=1}^{\infty} a_i \right| < \epsilon$.

When we consider this, we might conclude that 'sum of a series' is not a bad name as we can get arbitrarily close to the limit by adding a large enough number of terms – so adding more terms beyond the N that takes you to within ϵ of the limit isn't going to give you much more if ϵ is sufficiently small! Indeed if the sum of the series exists, it is common for even the most rigorous mathematicians to write

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots \quad (6.3.5)$$

as though we really do have an infinite process of addition going on! There's no harm in doing this as long as you appreciate that (6.3.5) is nothing more than a suggestive notation. The truth is in the ϵ s and N s of the limit concept.

If (s_n) diverges to $+\infty$ we sometimes use the notation $\sum_{i=1}^{\infty} a_i = \infty$. Similarly we write $\sum_{i=1}^{\infty} a_i = -\infty$ when (s_n) diverges to $-\infty$. Also (and I hope this terminology won't confuse anyone), if I write that $\sum_{i=1}^n a_i$ (or even $\sum_{i=1}^{\infty} a_i$) converges (or diverges) this is sometimes just a convenient shorthand for the convergence (or divergence) of (s_n) where $s_n = \sum_{i=1}^n a_i$. So when we talk of *convergent* or (*divergent*) *series*, we really mean the convergence (or divergence) of the associated sequence of partial sums.

⁴ Just to confuse you, many textbooks write $\sum_{i=1}^{\infty} a_i$ as $\sum_{n=1}^{\infty} a_n$ which is perfectly OK as i and n are dummy indices, but which might be a little strange at first because of the role we have given to n . For this reason I'm sticking to i , at least in this chapter where the ideas are new to you!

Now what about some examples? To make things simpler at the beginning, for the next two sections we'll focus on *nonnegative series*, i.e. those for which $a_i \geq 0$ for each i – so for example (6.1.1) is included, but not (6.1.2).

6.4 Nonnegative Series

OK – so far all we've really done is to give a definition and introduce some new notation. Now it's time to get down to the serious business of real mathematics. I'll remind you that for the next few sections we're going to focus on sequences (a_n) where each $a_n \geq 0$. We consider the partial sums (s_n) . Our first observation is that this is a monotonic increasing sequence as

$$s_{n+1} - s_n = \sum_{i=1}^{n+1} a_i - \sum_{i=1}^n a_i = \sum_{i=1}^n a_i + a_{n+1} - \sum_{i=1}^n a_i = a_{n+1} \geq 0,$$

and so $s_{n+1} \geq s_n$ for all n . It then follows from Corollary 5.2.1 that (s_n) either converges (to its supremum) or diverges to $+\infty$. Now we'll study some examples.

We've seen already that $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$ and this clearly diverges to $+\infty$. If you look at higher powers of i then their partial sums are even larger and so we should expect divergence again, e.g. $\sum_{i=1}^n i^2 = 1 + 2^2 + 3^2 + \dots + n^2 > n^2 > n$, and since (n) diverges, then so does $\sum_{i=1}^n i^2$. Indeed this is a consequence of Theorem 4.1.3.

A similar argument applies to $\sum_{i=1}^n i^m$ where $m > 1$ is any real number. What about $m = 0$? Well here we have $\sum_{i=1}^n i^0 = \sum_{i=1}^n 1 = n$ which again diverges. Finally if $0 < m < 1$, we have $i^m \geq 1$ and so $\sum_{i=1}^n i^m \geq \sum_{i=1}^n 1 = n$ which also diverges. So

we can conclude that $\sum_{i=1}^n i^m$ diverges to $+\infty$ for all $m \geq 0$. What happens when $m < 0$? Let's start by looking at the case $m = -1$. This is the famous *harmonic series* which is obtained by summing the terms of the harmonic sequence that we discussed in Section 4.1:⁵

$$\sum_{i=1}^n \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}.$$

⁵ For the relationship with the notion of harmonic in music see e.g. [http://en.wikipedia.org/wiki/Harmonic_series_\(music\)](http://en.wikipedia.org/wiki/Harmonic_series_(music))

A reasonable conjecture might be that this series converges as $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ so as n gets very large the difference between s_n and s_{n+1} is getting smaller and smaller. Indeed if we calculate the first few terms we find that $s_1 = 1$, $s_2 = \frac{3}{2}$, $s_3 = \frac{11}{6}$, $s_4 = \frac{25}{12}$, $s_5 = \frac{137}{60}$, \dots , so we might well believe that the sum of the series is a number between 2 and 3. But (as we should know by now), looking at the first five terms (or even the first billion) may not be a helpful guide to understanding convergence. In fact $\sum_{i=1}^n \frac{1}{i}$ diverges. To see why this is so, we'll employ a clever argument that collects terms together in powers of 2. We look at,⁶

$$\begin{aligned} \sum_{i=1}^{2^n} \frac{1}{i} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} \right) \\ &\quad + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) \\ &\quad + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \right) \\ &\quad + \dots \\ &\quad + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \dots + \frac{1}{2^n} \right) \dots (*) \end{aligned}$$

Now observe that

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2},$$

$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > 8 \cdot \frac{1}{16} = \frac{1}{2},$$

and we continue this argument until we get to

$$\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \dots + \frac{1}{2^n} = \frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \dots + \frac{1}{2^{n-1}+2^{n-1}}.$$

There are 2^{n-1} terms in this general sum and each term is greater than $\frac{1}{2^n}$ so

$$\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \dots + \frac{1}{2^n} > 2^{n-1} \cdot \frac{1}{2^n} = \frac{1}{2}.$$

If we count the number of brackets on the right-hand side of (*) and also include the terms 1 and $\frac{1}{2}$, we find that we have $(n+1)$ terms altogether and we have seen that n of these terms exceed $\frac{1}{2}$.

⁶ This argument is due to Nicole Oresme (1323?-1382), a Parisian thinker who eventually became Bishop of Lisieux - see e.g. http://en.wikipedia.org/wiki/Nicole_Oresme

We conclude that $\sum_{i=1}^{2^n} \frac{1}{i} > 1 + n \cdot \frac{1}{2} = 1 + \frac{n}{2}$ and we can see from this that the series diverges. Indeed given any $K > 0$ we can find n_0 such that $1 + \frac{n}{2} > K$ for all $n > n_0$ (just take n_0 to be the smallest natural number larger than $2(K - 1)$) and then $\sum_{i=1}^n \frac{1}{i} > K$ for all $n > 2^{n_0}$.

At this stage you might be getting the feeling that all series are divergent. You can rest assured that that is far from the case. There are plenty of convergent series around as we'll soon see. The next series on our list that we should consider is $\sum_{i=1}^n \frac{1}{i^2}$, but we need a few more tools before we can investigate that one. In fact we'll need to know about the related series $\sum_{i=1}^n \frac{1}{i(i+1)}$ and this will furnish our first example of a convergent series.

Example 6.1: $\sum_{i=1}^n \frac{1}{i(i+1)}$

To show this series converges we'll first find a neat formula for the n th partial sum. To begin with, you should check by cross-multiplication that

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

Next we write $\sum_{i=1}^n \frac{1}{i(i+1)}$ as a 'telescopic sum':⁷

$$\begin{aligned} \sum_{i=1}^n \frac{1}{i(i+1)} &= \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots \\ &\quad + \left(\frac{1}{n-2} - \frac{1}{n-1} \right) + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1}, \end{aligned}$$

after cancellation. So we conclude that

$$\sum_{i=1}^n \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}.$$

⁷ So called because the series can be compressed into a simple form by cancellation. The analogy is with the collapse of an old-fashioned telescope.

However, we know that the sequence whose n th term is $\frac{1}{n+1}$ converges to 0 and so we find that

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{i(i+1)} = 1 - \lim_{n \rightarrow \infty} \frac{1}{n+1} = 1.$$

This is our first successful encounter with a convergent series so we should allow ourselves a quick pause for appreciation. In this case we also have an example where the sum of the series is explicitly known. This is in fact quite rare. In most cases where we can show that a series converges we will not know the limit explicitly.

At this stage it is worth thinking a little bit about the relationship between the sequences (a_n) and (s_n) from the point of view of convergence. The last two examples we've considered are $\sum_{i=1}^n \frac{1}{i}$ and $\sum_{i=1}^n \frac{1}{i(i+1)}$. In the first of these we have $a_n = \frac{1}{n}$ and we know that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but we've shown that (s_n) diverges. In the second example, $a_n = \frac{1}{n(n+1)}$ and again we have $\lim_{n \rightarrow \infty} \frac{1}{n(n+1)} = 0$, but in this case (s_n) converges as we've just seen. It's time for a theorem:

Theorem 6.4.1. If $\sum_{i=1}^n a_i$ converges then so does the sequence (a_n) and $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. Suppose that $\lim_{n \rightarrow \infty} s_n = l$ then we also have $\lim_{n \rightarrow \infty} s_{n-1} = l$ (think about it). Now since

$$a_n = s_n - s_{n-1},$$

for all $n \geq 2$, we can use the algebra of limits, firstly to deduce that (a_n) converges and secondly to find that

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} \\ &= l - l = 0, \end{aligned}$$

and we are done. □

It's important to appreciate what Theorem 6.4.1 is really telling us. It says that *if* (s_n) converges *then* it follows that (a_n) converges to zero. It should not be confused with the converse statement: '*if* (a_n) converges to zero *then* it follows that (s_n) converges' which is false – and the case $a_n = \frac{1}{n}$ provides a counterexample. On the other hand, one of the most useful applications of Theorem 6.4.1 is to prove that a series diverges, for if we use the fact that a statement is logically equivalent to its contrapositive,⁸ we see that Theorem 6.4.1 also tells us that if (a_n)

⁸ The *contrapositive* of 'If P then Q,' is 'If not Q then not P'.

does not converge to zero then $\sum_{i=1}^n a_i$ diverges, e.g. we see immediately that $\sum_{i=1}^n \frac{i}{i+100}$ diverges since $\lim_{n \rightarrow \infty} \frac{n}{n+100} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{100}{n}} = 1$ by algebra of limits.

By the way, we said that we'd only consider nonnegative series in this section, but you can check that Theorem 6.4.1 holds without this constraint.

6.5 The Comparison Test

The theory of series is full of *tests for convergence* which are various tricks that have been developed over the years for showing that a series is convergent or divergent. We'll meet a small number of these in this chapter. In fact we can't proceed further in our quest to show convergence of $\sum_{i=1}^n \frac{1}{i^2}$ without being able to use the comparison test, and we'll present this as a theorem. The proof of (1) is particularly sweet as it makes use of old friends from earlier chapters.

Theorem 6.5.1 (The Comparison Test). Suppose that (a_n) and (b_n) are sequences with $0 \leq a_n \leq b_n$ for all n . Then

1. if $\sum_{i=1}^n b_i$ converges then so does $\sum_{i=1}^n a_i$,
2. if $\sum_{i=1}^n a_i$ diverges then so does $\sum_{i=1}^n b_i$.

Proof. Throughout this proof we'll write $s_n = \sum_{i=1}^n a_i$ (as usual) and $t_n = \sum_{i=1}^n b_i$.

1. We are given that the sequence (t_n) converges and so it is bounded by Theorem 4.2.1. In particular it is bounded above and since each t_n is positive, it follows that there exists $K > 0$ such that $t_n = |t_n| \leq K$ for all n . Now since each $a_i \leq b_i$ we have for all n that

$$s_n = \sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i = t_n \leq K,$$

and so the sequence (s_n) is bounded above. We have already pointed out that (s_n) is monotonic increasing and so we can apply Theorem 5.2.1 (1) to conclude that (s_n) converges as required.

2. This is really just a special case of Theorem 4.1.3, but since I didn't prove that result I will do so for this one. The sequence (s_n) diverges so given any $L > 0$ there exists a natural number n_0 such that $s_n > L$ whenever $n > n_0$. But since each $b_i \geq a_i$ we can argue as in (1) to deduce that $t_n \geq s_n > L$ for such n and hence (t_n) diverges. \square

We'll now (as promised) apply the comparison test to show that $\sum_{i=1}^n \frac{1}{i^2}$ converges.

Example 6.2: $\sum_{i=1}^n \frac{1}{i^2}$

We know from Example 6.1 that $\sum_{i=1}^n \frac{1}{i(i+1)}$ converges and so by the algebra of limits, so does $2\sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n \frac{2}{i(i+1)}$.

Now for all natural numbers i ,

$$\begin{aligned} \frac{2}{i(i+1)} - \frac{1}{i^2} &= \frac{1}{i} \left[\frac{2}{i+1} - \frac{1}{i} \right] \\ &= \frac{1}{i} \left[\frac{2i - (i+1)}{i(i+1)} \right] \\ &= \frac{i-1}{i^2(i+1)} \geq 0. \end{aligned}$$

So if we take $a_i = \frac{1}{i^2}$ and $b_i = \frac{2}{i(i+1)}$, we have $0 \leq a_i \leq b_i$ and so by Theorem 6.5.1 (1), $\sum_{i=1}^n \frac{1}{i^2}$ converges.

We've now shown that $\sum_{i=1}^n \frac{1}{i^2}$ converges. But is it possible to find an exact value for the sum of this series? The problem was first posed by Jacob Bernoulli (1654–1705).⁹ As Bernoulli was living in Basle, Switzerland at the time the problem of finding a real number k such that $\sum_{i=1}^{\infty} \frac{1}{i^2} = k$ became known as the 'Basle problem'. The problem was solved by one of the greatest mathematicians of all time,

⁹ He was one of three brothers who all made important mathematical contributions. Furthermore, the sons and grandsons of this remarkable trio produced another five mathematicians – see e.g. http://en.wikipedia.org/wiki/Bernoulli_family. Be aware that Jacob is sometimes called by his Anglicised name James and should not be confused with his younger brother Johann (also called John).

Leonhard Euler¹⁰ (1706–90) in 1735. He showed that

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6},$$

so that $k = \frac{\pi^2}{6}$ which is 1.6449 to four decimal places. This connection between the sums of inverses of squares of natural numbers and π – the universal constant which is the ratio of the circumference of any circle to its diameter – is really beautiful and may appear a little mysterious. To give Euler's original proof goes beyond the scope of this book.¹¹ If you take a university level course that teaches you the idea of a Fourier series then there is a very nice succinct proof which uses that concept in a delightful way.

Now that we have the comparison test up our sleeves, then we can make much more progress in our goal to fully understand when $\sum_{i=1}^n \frac{1}{i^r}$ converges for various values of r .

Example 6.3: $\sum_{i=1}^n \frac{1}{i^r}$ for $r > 2$.

All the series of this type converge. To see this it's enough to notice that if i is any natural number then whenever $r > 2$

$$i^r \geq i^2.$$

Now by (L5) we have

$$\frac{1}{i^r} \leq \frac{1}{i^2},$$

and we can immediately apply the comparison test (Theorem 6.5.1 (1)) to deduce that $\sum_{i=1}^n \frac{1}{i^r}$ converges for $r > 2$.

Example 6.4: $\sum_{i=1}^n \frac{1}{i^r}$ for $0 < r < 1$.

In this case we have $i^r < i$ for all natural numbers i and so again by (L5), $\frac{1}{i} < \frac{1}{i^r}$. Now apply the comparison test in the form Theorem 6.5.1 (2) to deduce that $\sum_{i=1}^n \frac{1}{i^r}$ diverges for $0 < r < 1$.

¹⁰ See http://en.wikipedia.org/wiki/Leonhard_Euler

¹¹ The best account of this that I've come across is in Jeffrey Stopple's superb textbook *A Primer of Analytic Number Theory*, Cambridge University Press (2003). See the Further Reading section for more about this book.

I've already told you how Euler found an exact formula for $\sum_{i=1}^{\infty} \frac{1}{i^2}$ which featured π^2 . He also discovered exact formulae for $\sum_{i=1}^{\infty} \frac{1}{i^r}$ where r is an *even number* and these are all expressed in terms of π^r . I won't write down the exact formulae here as they are more complicated than the case $r = 2$. We'll postpone that to the next chapter as there is a fascinating connection with the number e which we're going to study there.¹² Remarkably, very little is still known about $\sum_{i=1}^{\infty} \frac{1}{i^r}$ in the case where r is an odd number ($r \geq 3$). We had to wait until 1978 for Roger Apéry (1916–94) to prove that $\sum_{i=1}^{\infty} \frac{1}{i^3}$ is an irrational number!

To complete the story of $\sum_{i=1}^n n^r$ where r is any real number, we should look at $\sum_{i=1}^n \frac{1}{i^r}$ for $1 < r < 2$. We'll need a new technique before we do that and this is the theme of the next section.

Before we do that, let's look at one more interesting series. We've shown that $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges. Now we'll consider the sum of all reciprocals of the *square-free integers*:

$$1, 2, 3, 5, 6, 7, 10, 11, 13, 14, \dots,$$

i.e. those natural numbers that can never have a perfect square as a factor.

It seems that this is a 'smaller sum' so it may be possible that it converges. We'll denote the generic square-free integer as i_{sf} where the suffix *sf* stands for 'square-free' and consider $\sum_{i_{sf}=1}^{\infty} \frac{1}{i_{sf}}$.¹³ Remember that in Chapter 1, we showed that any natural number n can be written as $n = i_{sf}m^2$ where $m \leq n$ is a natural number.

Theorem 6.5.2. $\sum_{i_{sf}=1}^{\infty} \frac{1}{i_{sf}}$ diverges.

Proof. The result more or less follows from the inequality

$$\sum_{i=1}^n \frac{1}{i} \leq \left(\sum_{i_{sf}=1}^n \frac{1}{i_{sf}} \right) \left(\sum_{m=1}^n \frac{1}{m^2} \right).$$

¹² If you can't wait, try looking in Section 6.2 of Stoppole's book which is cited in footnote 11.

¹³ It may be that a better notation for this is $\sum_{i_{sf} < \infty} \frac{1}{i_{sf}}$, which we define to mean precisely

$$\lim_{n \rightarrow \infty} \sum_{i_{sf} < n} \frac{1}{i_{sf}}.$$

To see that this holds you need only use the fact that for any $1 \leq i \leq n$, $\frac{1}{i} = \frac{1}{i_{sf}} \frac{1}{m^2}$ where $m \leq n$ and $i_{sf} \leq n$, so $\frac{1}{i}$ certainly appears in the product of the sums on the right-hand side – and that is enough. We've seen that $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges (to a positive number which is the supremum of the sequence of partial sums) and so we can make the right-hand side of the last inequality larger to obtain

$$\sum_{i=1}^n \frac{1}{i} \leq \left(\sum_{i_{sf}=1}^n \frac{1}{i_{sf}} \right) \left(\sum_{m=1}^{\infty} \frac{1}{m^2} \right) = k \sum_{i_{sf}=1}^n \frac{1}{i_{sf}},$$

where $k = \sum_{m=1}^{\infty} \frac{1}{m^2}$ (which we pointed out earlier is in fact equal to $\frac{\pi^2}{6}$).

From this and Theorem 4.1.3 we see that the divergence of $\sum_{i_{sf}=1}^{\infty} \frac{1}{i_{sf}}$ follows from that of $\sum_{i=1}^{\infty} \frac{1}{i}$. □

Corollary 6.5.1. There are infinitely many square-free integers.

Proof. Suppose by way of contradiction that there was a largest square-free integer I_{sf} . Then

$$\sum_{i_{sf}=1}^{\infty} \frac{1}{i_{sf}} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{I_{sf}},$$

and this is a finite sum of numbers. This contradicts the result of Theorem 6.5.2. □

Since all prime numbers are square-free you may wonder whether the sum of all $\frac{1}{p}$ (where p is prime) converges or diverges. We'll come back to settle that question in Chapter 8.

6.6 Geometric Series

In this section we'll look at series such as

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots, \tag{6.6.6}$$

$$6 + 18 + 54 + 162 + 486 + \cdots \tag{6.6.7}$$

Both of these are examples of *geometric series*, i.e. they are of the form $\sum_{i=0}^{\infty} ar^i = a + ar + ar^2 + ar^3 + \cdots$. The number a is (surprise, surprise) called the

first term and r is called the *common ratio*.¹⁴ So in (6.6.6) $a = 1$ and $r = \frac{1}{2}$. In (6.6.7), $a = 6$ and $r = 3$. You might have guessed that (6.6.7) diverges and speculated that (6.6.6) may well converge. To investigate this in the general case we'll first find a general formula for the n th partial sum $s_n = \sum_{i=0}^n ar^i$.¹⁵ This finite series is sometimes called a *geometric progression*. To find the general formula we first note that if $r = 1$ then

$$s_n = a + a + \cdots + a = (n + 1)a. \quad (6.6.8)$$

Now assume that $r \neq 1$ to find that

$$s_n = a + ar + ar^2 + \cdots + ar^n. \quad (6.6.9)$$

Then multiply both sides of (6.6.9) by r to obtain

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}. \quad (6.6.10)$$

Notice that all the terms on the right-hand sides of (6.6.9) and (6.6.10) are common to both equations, except the first term a in (6.6.9) and the last term ar^{n+1} in (6.6.10). If we subtract (6.6.10) from (6.6.9) we obtain

$$(1 - r)s_n = a - ar^{n+1},$$

and since $r \neq 1$, it is legitimate to divide both sides by $1 - r$ to get the formula we are seeking:

$$s_n = \frac{a(1 - r^{n+1})}{1 - r}. \quad (6.6.11)$$

For example, you can use this to quickly calculate the sum of the first 10 terms in (6.6.6) by spotting that $a = 1$ and $r = \frac{1}{2}$ in this case. So we want

$$s_9 = \frac{1 \cdot \left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}} = 2 - \frac{1}{2^9} = 2 - \frac{1}{512} = \frac{1023}{512}.$$

You can use a similar argument to deduce that $s_n = 2 - \frac{1}{2^n}$ and this sequence clearly converges to 2.

More generally if $|r| < 1$ (so $-1 < r < 1$) then, by Example 4.2, we have $\lim_{n \rightarrow \infty} r^n = 0$. Hence if we apply the algebra of limits in (6.6.11) we see that the geometric series converges and

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1 - r}. \quad (6.6.12)$$

¹⁴ To understand why the word 'geometric' is used here go to the section "Relationship to geometry and Euclid's work" in http://en.wikipedia.org/wiki/Geometric_progression

¹⁵ Note that we start at $i = 0$ here instead of $i = 1$, but this introduces no new difficulties in finding limits. If you want to you can even systematically replace $\sum_{i=0}^n ar^i$ with $\sum_{i=1}^{n+1} ar^{i-1}$. However do bear in mind that s_n is now the sum of the first $n + 1$ terms.

In fact it is easy to check using (6.6.8) and (6.6.11) that the geometric series converges for no other value of r . Indeed if $r \geq 1$ it diverges to $+\infty$, if $r = -1$ it oscillates finitely between 0 and a and if $r < -1$ it oscillates infinitely.

Now we are going to use the geometric series as a tool in proving that $\sum_{i=1}^n \frac{1}{i^r}$ converges for $1 < r < 2$. We'll use a similar trick to the one we employed to prove that $\sum_{i=1}^n \frac{1}{i}$ diverges. We'll write the natural numbers in the form

$$2^0, 2^0 + 1, 2^1, 2^1 + 1, 2^2, 2^2 + 1, 2^2 + 2, 2^3 - 1, 2^3, \dots$$

Now consider all the terms written in this form that lie between 2^{i-1} and $2^i - 1$ (including these two 'end-points') where i is an arbitrary natural number. There are exactly 2^{i-1} such numbers (think about it – and look at the case $i = 3$ from the list above for inspiration if necessary). Now define

$$b_i = \frac{1}{(2^{i-1})^r} + \frac{1}{(2^{i-1} + 1)^r} + \frac{1}{(2^{i-1} + 2)^r} + \dots + \frac{1}{(2^i - 1)^r}.$$

Since $\frac{1}{(2^{i-1})^r}$ is the largest number which appears on the right-hand side we get

$$b_i \leq 2^{i-1} \cdot \frac{1}{(2^{i-1})^r} = \left(\frac{1}{2^{i-1}}\right)^{r-1} = \left(\frac{1}{2^{r-1}}\right)^{i-1},$$

and so $\sum_{i=1}^n b_i \leq \sum_{i=1}^n \left(\frac{1}{2^{r-1}}\right)^{i-1}$.

But the series on the right-hand side is a geometric one having first term 1 and common ratio $\frac{1}{2^{r-1}} < 1$ since $r > 1$. So this geometric series converges and hence by the comparison test so does $\sum_{i=1}^n b_i$. But this series is nothing but $\sum_{i=1}^n \frac{1}{i^r}$ rewritten in a clever way that involves powers of 2. And that's it.¹⁶

We have now completed the story of $\sum_{i=1}^{\infty} i^r$ where r is a real number. We have shown that the series diverges if $r \geq -1$ and converges if $r < -1$. The 'parameter' r plays a similar role here to the temperature of substances like water. Water is frozen solid for temperatures below 0 degrees celsius but at that temperature it melts to form a liquid and this is called a 'phase transition' in physics. If we keep on increasing the temperature then the water stays liquid until we get to 100 degrees celsius, when it changes to a gas and this is a second phase transition. By analogy we can regard the value $r = -1$ as indicating a phase transition between the regions where the series $\sum_{i=1}^n i^r$ diverges and converges (respectively). This analogy

¹⁶ In fact this is an example of *regrouping* a series (see Section 6.10) and to be completely rigorous, Theorem 6.10.1 should precede the argument we've just given.

may not be as far-fetched as it seems as analysis plays an important role in the mathematical modelling of real phase transitions.

6.7 The Ratio Test

In the last section we saw the benefits of the comparison test for proving convergence of series. Although it is a wonderful thing, it is by no means the only tool that we need for playing the series game and indeed there are many important series such as $\sum_{i=1}^{\infty} \frac{1}{i!}$ where it doesn't help at all.¹⁷ In this section we'll develop another useful test called the *ratio test*.¹⁸ Before we prove this, it will be helpful to make some general remarks about infinite series.

Suppose that we have a finite series $\sum_{i=1}^n a_i$ where each $a_i \geq 0$. We can split the sum at any intermediate point

$$\sum_{i=1}^n a_i = \sum_{i=1}^m a_i + \sum_{i=m+1}^n a_i, \quad (6.7.13)$$

Can we do the same for infinite sums? Suppose that $\sum_{i=1}^n a_i$ converges to l . Now consider the sequence whose n th term is $\sum_{i=m+1}^n a_i$. This series also converges by the comparison test as we have (using the notation of Theorem 6.5.1) $b_i \leq a_i$ for each i where $b_i = 0$ if $1 \leq i \leq m$ and $b_i = a_i$ if $i \geq m + 1$. It is then natural to define

$$\sum_{i=m+1}^{\infty} a_i = \lim_{n \rightarrow \infty} \sum_{i=m+1}^n a_i.$$

Applying the algebra of limits in (6.7.13) we find that

$$\begin{aligned} \sum_{i=m+1}^{\infty} a_i &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n a_i - \sum_{i=1}^m a_i \right) \\ &= \sum_{i=1}^{\infty} a_i - \sum_{i=1}^m a_i, \end{aligned}$$

¹⁷ Remember that $i! = i(i-1)(i-2) \cdots 3 \cdot 2 \cdot 1$.

¹⁸ Sometimes called *d'Alembert's ratio test* in honour of the French thinker Jean d'Alembert (1717–83) who was the first to publish it – see http://en.wikipedia.org/wiki/Jean_le_Rond_d'Alembert

and so

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^m a_i + \sum_{i=m+1}^{\infty} a_i. \quad (6.7.14)$$

Notice from this that to show $\sum_{i=1}^{\infty} a_i$ converges it's enough to prove that $\sum_{i=m+1}^{\infty} a_i$ converges for any fixed m which can be as large as we like. This makes sense from an intuitive point of view as it's the 'tail' of the series, i.e. it's value beyond a certain point, that determines convergence.

We're now ready to describe the ratio test and as usual we'll state the test as a theorem. The proof of this gives us another opportunity to appreciate the value of geometric series.

Theorem 6.7.1 (The Ratio Test). Suppose that (a_n) is a sequence of positive real numbers for which $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l$, then

- $\sum_{i=1}^n a_i$ converges if $l < 1$.
- $\sum_{i=1}^n a_i$ diverges if $l > 1$.
- If $l = 1$ the test is inconclusive and the series may converge or diverge.

Proof. First suppose that $l < 1$ and notice that we can then find an $\epsilon > 0$ such that

$$l + \epsilon < 1 \dots (i),$$

indeed no matter how close to 1 the real number l may get, there is always some gap and choosing $\epsilon = \frac{1}{2}(1 - l)$, for example, will only fill half of that gap.

Now let's go back to the definition of the limit of a sequence. Since $\left(\frac{a_{n+1}}{a_n}\right)$ converges to l and given the value of ϵ that we've just chosen to satisfy (i), we know there exists a natural number n_0 such that if $n > n_0$ then $\left|\frac{a_{n+1}}{a_n} - l\right| < \epsilon$, i.e.

$$l - \epsilon < \frac{a_{n+1}}{a_n} < l + \epsilon \dots (ii)$$

Now we can write each

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \dots \frac{a_{n_0+2}}{a_{n_0+1}} a_{n_0+1}.$$

Each ratio on the right-hand side satisfies (ii) and so we have

$$a_n < (l + \epsilon)^{n-n_0-1} a_{n_0+1},$$

since there are precisely $n - n_0 - 1$ ratios on the right-hand side. Now a_{n_0+1} is a fixed number and $\sum_{n=n_0+1}^{\infty} (l + \epsilon)^{n-n_0-1} a_{n_0+1}$ is a geometric series with first term a_{n_0+1} and common ratio $l + \epsilon$. This converges by (i) and so by the comparison test (bearing in mind (6.7.14)), $\sum_{n=1}^{\infty} a_n$ converges.

Now suppose that $l > 1$ and choose $\epsilon = l - 1$ in the left-hand side inequality of (ii). Then we can find m_0 such that if $m > m_0$ then $\frac{a_{m+1}}{a_m} > 1$, i.e. $a_{m+1} > a_m$. But then we cannot possibly have $\lim_{m \rightarrow \infty} a_m = 0$ and so $\sum_{n=1}^{\infty} a_n$ diverges by Theorem 6.4.1.¹⁹

To see that anything can happen when $l = 1$ consider how $\sum_{i=1}^n a_i$ behaves as $n \rightarrow \infty$ in the two cases $a_n = \frac{1}{n}$ and $a_n = \frac{1}{n^2}$. □

Note that Theorem 6.7.1 assumes implicitly that we are dealing with a series for which $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ exists. If it doesn't then we cannot apply the ratio test (at least not this form of it, see Exercise 6.11 for a more general version).

Example 6.5: $\sum_{i=1}^{\infty} \frac{x^i}{i!}$

We'll use the ratio test to examine the convergence of the series $\sum_{i=1}^n \frac{x^i}{i!}$ where x is an arbitrary positive number. This series will play an important role in the next chapter when we'll be learning about the irrational number that is denoted by e . The ratio test is easy to apply in this case, we have

$$a_n = \frac{x^n}{n!}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)!},$$

and so

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x \cdot n!}{(n+1)n!} = \frac{x}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

irrespective of the value of x . So $l = 0 < 1$ and we conclude that the series converges for all values of x . This proof has given us a lot. Not only do we know that $\sum_{i=1}^n \frac{1}{i!}$ converges (in fact, as we'll see in Chapter 7, the sum of the series is the special number e) but also that, e.g. $\sum_{i=1}^n \frac{102039^i}{i!}$ converges.

¹⁹ Recall that I told you that this result would be helpful in proving divergence.

6.8 General Infinite Series

So far in this chapter we've only considered infinite series that have nonnegative terms. But what about more general series such as $\sum_{i=1}^n (-1)^i$ or $\sum_{i=1}^n \frac{x^i}{i!}$ where x is an arbitrary real number? Let's focus on the first of these. If we group the terms as

$$(-1 + 1) + (-1 + 1) + (-1 + 1) + \dots$$

then it seems to be converging to zero but if we write it as

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots$$

it looks like it converges to 1. But in fact this series diverges by Theorem 6.4.1. We should take this as a 'health warning' that dealing with negative numbers in infinite series might lead to headaches. We'll come back to *regrouping* terms in series later in this chapter. Now let's focus on the general picture. We are interested in the convergence (or otherwise) of a series $\sum_{i=1}^n a_i$ where (a_n) is an arbitrary sequence of real numbers, so we've dropped the constraint that these numbers are all nonnegative. It's a shame to lose all the knowledge we've gained in the early part of this chapter so let's introduce a link to that material. To each general series $\sum_{i=1}^n a_i$ we can associate the nonnegative series $\sum_{i=1}^n |a_i|$. Here's a key definition. The series $\sum_{i=1}^n a_i$ is said to be *absolutely convergent* if $\sum_{i=1}^n |a_i|$ converges. So, for example, the series $\sum_{i=1}^n (-1)^{i+1} \frac{1}{i^2}$ is absolutely convergent since $\sum_{i=1}^n |(-1)^{i+1} \frac{1}{i^2}| = \sum_{i=1}^n \frac{1}{i^2}$ converges. Now all we've done so far is make a definition. The next theorem tells us why this is useful.

Theorem 6.8.1. Any absolutely convergent series is convergent.

Proof. We want to show that the sequence whose n th term is $s_n = \sum_{i=1}^n a_i$ converges. Suppose that it is absolutely convergent. Then the sequence (t_n) converges where $t_n = \sum_{i=1}^n |a_i|$. Now each $|a_i| \geq a_i$ and $|a_i| \geq -a_i$ (recall Section 3.4) and so

$$0 \leq a_i + |a_i| \leq 2|a_i|.$$

By algebra of limits, $2\sum_{i=1}^n |a_i|$ converges and hence by the comparison test so does $u_n = \sum_{i=1}^n (a_i + |a_i|)$. Then by algebra of limits again we have convergence of $s_n = u_n - t_n$ and our job is done. \square

By Theorem 6.8.1 we see immediately that $\sum_{i=1}^n (-1)^{i+1} \frac{1}{i^2}$ converges. Next we have an important example that picks up an earlier theme.

Example 6.6: $\sum_{i=1}^{\infty} \frac{x^i}{i!}$

We can now show that this series converges when x is an arbitrary real number. Indeed we've already shown this when $x \geq 0$. If $x < 0$ then

$$\sum_{i=1}^n \left| \frac{x^i}{i!} \right| = \sum_{i=1}^n \frac{|x|^i}{i!} = \sum_{i=1}^n \frac{|x|^i}{i!}$$

and since $|x| \geq 0$ this last series converges. So $\sum_{i=1}^n \frac{x^i}{i!}$ is absolutely convergent and hence is convergent by Theorem 6.8.1.

Both of the tests for convergence that we've met can be souped up into tests for general series. I'll state these but if you want proofs, you'll have to provide the details (see Exercise 6.12). Both cases are quite straightforward to deal with.

The Comparison Test – general case. If (a_n) is an arbitrary sequence of real numbers and (b_n) is a sequence of nonnegative numbers so that $|a_n| < b_n$ for all n then if $\sum_{i=1}^n b_n$ converges so does $\sum_{i=1}^n a_i$.

The Ratio Test – general case. If (a_n) is a sequence of real numbers for which $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = l$, then if $l < 1$, $\sum_{i=1}^n a_i$ converges, if $l > 1$, $\sum_{i=1}^n a_i$ diverges and if $l = 1$ then the test is inconclusive.

6.9 Conditional Convergence

We've seen in the last section that every absolutely convergent series is convergent. In this section we'll focus on the convergence of series that may not be absolutely convergent. For example, consider the series $\sum_{i=1}^n (-1)^{i+1} \frac{1}{i}$ which begins $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$. This series is certainly not absolutely convergent

as we've already shown that the harmonic series $\sum_{i=1}^n \frac{1}{i}$ diverges. But the partial sums of this series will certainly be smaller than those of the harmonic series, so perhaps there is a chance that it will converge. Before we investigate further we'll need another definition. If (a_n) is a sequence of real numbers for which $\sum_{i=1}^n a_i$ converges but $\sum_{i=1}^n |a_i|$ diverges, then the series $\sum_{i=1}^n a_i$ is said to be *conditionally convergent*. We've not met any conditionally convergent series yet but the next theorem, which gives us another test for convergence, will give us the tool we need to find them. This convergence test is named after Gottfried Leibniz (1646–1716),²⁰ who was a renaissance man, par excellence! In his well-known book *Men of Mathematics*²¹ that gives a series of short biographies of famous (male) mathematicians, E.T. Bell writes, 'Mathematics was one of the many fields in which Leibniz showed conspicuous genius: law, religion, statecraft, history, logic, metaphysics and speculative philosophy all owe to him contributions, any one of which would have secured his fame and preserved his memory'.

Theorem 6.9.1 (Leibniz' Test). Let (a_n) be a sequence of nonnegative numbers that is

- (a) monotonic decreasing, (b) convergent to zero.

In this case the series $\sum_{i=1}^n (-1)^{i+1} a_i$ converges.

Proof. Let $s_n = \sum_{i=1}^n (-1)^{i+1} a_i$. Let n be even so that $n = 2m$ for some m . Then

$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}).$$

It follows that

$$s_{2m+2} = s_{2(m+1)} = s_{2m} + (a_{2m+1} - a_{2m+2}) \geq s_{2m},$$

since (a_n) is monotonic decreasing. This shows that the sequence (s_{2n}) is monotonic increasing. It is also bounded above since

$$\begin{aligned} s_{2m} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m} \\ &\leq a_1. \end{aligned}$$

Here we've used the fact that the sequence (a_n) is nonnegative so that $a_{2m} \geq 0$ and that it is also monotonic decreasing, so each bracket is nonnegative. We can

²⁰ See http://en.wikipedia.org/wiki/Gottfried_Leibniz

²¹ This famous book was first published in 1937. The most recent edition is in Touchstone Press (Simon and Schuster Inc.) 1986.

now apply Theorem 5.2.1 (1) to conclude that (s_{2m}) converges and we define $s = \lim_{m \rightarrow \infty} s_{2m}$. We now know that even partial sums converge. What about the odd ones? Well by algebra of limits we have

$$\lim_{m \rightarrow \infty} s_{2m+1} = \lim_{m \rightarrow \infty} s_{2m} + \lim_{m \rightarrow \infty} a_{2m+1} = s + 0 = s.$$

We've shown that $\lim_{m \rightarrow \infty} s_{2m} = s$ and $\lim_{m \rightarrow \infty} s_{2m+1} = s$. To prove the theorem we need to show that the full sequence (s_n) converges. It seems feasible that if it does, then $\lim_{n \rightarrow \infty} s_n = s$ and this is what we'll now prove. It's about time we had an ϵ and an n_0 again, so let's fix $\epsilon > 0$. Then from what has been proved above, there exists n_0 such that if $m > n_0$ then $|s_{2m} - s| < \epsilon$ and there exists n_1 such that if $m > n_1$ then $|s_{2m+1} - s| < \epsilon$. Now let $n > \max(n_0, n_1)$. Then either n is even and so $n = 2m$ for some m , or n is odd in which case $n = 2m + 1$. In either case we have $|s_n - s| < \epsilon$ and the result is proved. \square

It is very easy to apply Leibniz' test to see immediately that $\sum_{i=1}^n (-1)^{i+1} \frac{1}{i}$ converges and this gives us a nice example of a conditionally convergent series. In fact this is an example of a series where the sum is known and it is $\log_e(2)$ (the logarithm to base e of 2 whose decimal expansion begins 0.6931471). The proof uses calculus which goes beyond the scope of this book but I've included a sketch below for those who know some integration (and as an incentive to learn about it for those who don't).

We start with the following binomial series expansion which is valid for $-1 < x < 1$:

$$\begin{aligned} (1+x)^{-1} &= \sum_{n=0}^{\infty} (-1)^n x^n \\ &= 1 - x + x^2 - x^3 + x^4 - \dots \end{aligned}$$

Now integrate both sides (the interchange of integration with infinite summation on the right-hand side needs justification) to get

$$\begin{aligned} \log_e(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}. \end{aligned}$$

Because of the constraint on x we can't just put $x = 1$ (tempting though this may be) but after some careful work, it turns out that you can take the *limit* as $x \rightarrow 1$ (from below) and this gives the required result.

6.10 Regrouping and Rearrangements

There are two ways in which we can mix up the terms in an infinite series – by regrouping and by rearrangement. In the first of these we add the series in a different way by bracketing the terms differently (I did this earlier for the series $\sum_{i=1}^n (-1)^i$) but we don't change the order in which terms appear. In the second, we mix up the order in which terms appear as much as we please. For example, consider the series $\sum_{i=1}^n \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$. An example of a regrouping of this series is

$$1 + \left(\frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{16} + \frac{1}{25} + \frac{1}{36}\right) + \dots = 1 + \frac{13}{36} + \frac{469}{3600} + \dots$$

and here is a rearrangement of the same series:

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{36} + \frac{1}{4} + \frac{1}{49} + \frac{1}{36} + \dots$$

If a series is convergent then regrouping can't do it much harm but rearrangements can wreak havoc as we will see. First let's look at regrouping:

Theorem 6.10.1. If a series converges to l then it continues to converge to the same limit after any regrouping.

Proof. Suppose the series $\sum_{i=1}^n a_i$ converges to l . We'll write a general regrouping of the series as follows:

$$\begin{array}{rcl} b_1 & = & a_1 + a_2 + \dots + a_{m_1} \\ b_2 & = & a_{m_1+1} + a_{m_2+2} + \dots + a_{m_2} \\ \cdot & \cdot & \dots \dots \\ \cdot & \cdot & \dots \dots \\ \cdot & \cdot & \dots \dots \\ b_n & = & a_{m_{n-1}+1} + a_{m_{n-1}+2} + \dots + a_{m_n}. \end{array}$$

Then $\sum_{i=1}^n b_i = \sum_{r=1}^{m_n} a_r$ and we have $m_n \geq n$. Since the original series converges we know that given $\epsilon > 0$ there exists n_0 such that if $m_n > n_0$ then $\left| \sum_{r=1}^{m_n} a_r - l \right| < \epsilon$. Now if $n > n_0$ we must have $m_n > n_0$ and so $\left| \sum_{i=1}^n b_i - l \right| < \epsilon$ which gives the required convergence. \square

The contrapositive of the statement of this theorem tells us that if a series converges to more than one limit after regrouping then it diverges and this gives us another proof that $\sum_{i=1}^n (-1)^n$ diverges, as we've already shown how to group it in such a way that it converges to 0 and to 1.

Rearrangements are more complicated. We won't prove anything about them here but will be content with just stating two results which are both originally due to the nineteenth-century German mathematician Bernhard Riemann (1826–66):²²

- If $\sum_{i=1}^n a_i$ is absolutely convergent to l then any rearrangement of the series is also convergent to the same limit.
- If $\sum_{i=1}^n a_i$ is conditionally convergent then given any real number x , it is possible to find a rearrangement such that the rearranged series converges to x . In fact rearrangements can even be found for which the series diverges to $+\infty$ or $-\infty$.

The second result quoted here is quite mindboggling and the two results taken together illustrate that there is quite a significant difference in behaviour between absolutely convergent and conditionally convergent series. To see a concrete example of how to rearrange conditionally convergent series to converge to different values look at pp. 177–8 of D.Bressoud *A Radical Approach to Real Analysis* (second edition), Mathematical Association of America (2007).²³

We'll close this section with a remark about divergent series. You may think that once a series has been shown to diverge then that's the end of the story. In fact it can sometimes make sense to assign a number to a divergent series and even refer to it as the 'sum' – where 'sum' is interpreted in a different way from the usual. For example consider a slight variation on our old friend (6.1.2) – the series $\sum_{i=1}^n (-1)^{i+1}$. The great Leonhard Euler noticed that this is a geometric series with first term 1 and common ratio -1 . Even though the formula (6.6.12) is not valid in this context, Euler applied it and argued that the series 'converges' to $\frac{1}{2}$. Euler's reasoning was incorrect here but his intuition was sound. If you redefine summation of a series to mean, taking the limit of averages of partial sums rather than partial sums themselves, then this is precisely the answer that you get. If you want to explore divergent series further from this point of view then a good place to start is Exercise 6.19. See also http://en.wikipedia.org/wiki/Divergent_series.

²² See http://en.wikipedia.org/wiki/Bernhard_Riemann

²³ This book is briefly discussed in the Further Reading section.

6.11 Real Numbers and Decimal Expansions

In this book we've adopted a working definition of a real number as one that has a decimal expansion. But what do we really mean by this? Since $a_0.a_1a_2a_3a_4\cdots = a_0 + 0.a_1a_2a_3a_4\cdots$ if the number $a_0.a_1a_2a_3a_4\cdots$ is positive and $a_0 - 0.a_1a_2a_3a_4\cdots$ if it is negative, we might as well just consider those real numbers that lie between 0 and 1 for the purposes of this discussion. Now

$$0.a_1a_2a_3a_4\cdots = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \frac{a_4}{10^4} + \cdots$$

so we can see that decimal expansions are really nothing but a convenient shorthand for convergent infinite series of the form $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$. As we've already remarked, a rational number either has a finite decimal expansion such as $\frac{1}{2} = \frac{5}{10} + \frac{0}{100} + \frac{0}{1000} + \cdots$ or the a_n s are given by a periodic (or eventually periodic) pattern such as $\frac{1}{3} = \sum_{n=1}^{\infty} \frac{3}{10^n}$ so each $a_n = 3$ in this example.

By the way, it appears that we have privileged the number 10 in this story but (as discussed in section 2.1) that is just a matter of convenience and collective habit. We could just as easily work in base 2 for example and represent all numbers between 0 and 1 as binary decimals $\sum_{n=1}^{\infty} \frac{b_n}{2^n}$. e.g. $\frac{1}{2} = 0.1$ in this base and $\frac{1}{3} = 0.\dot{0}1 = \sum_{n=1}^{\infty} \frac{1}{4^n}$. This is of course a geometric series and you should check that it has the right limit. We stick to base 10 from now on because we're used to it (recall the discussion in Section 2.1).

An interesting phenomenon occurs with numbers whose decimal expansion is always 9 after a given point, so they look like $x = a_1a_2\cdots a_N999\cdots = a_1a_2\cdots a_N\dot{9} = a_1a_2\cdots a_N + \sum_{r=N+1}^{\infty} \frac{9}{10^r}$.

Let's focus on $\sum_{r=N+1}^{\infty} \frac{9}{10^r}$. This is a geometric series whose first term is $\frac{9}{10^{N+1}}$ and common ratio is $\frac{1}{10}$. So it converges to

$$\frac{\frac{9}{10^{N+1}}}{1 - \frac{1}{10}} = \frac{\frac{9}{10^{N+1}}}{\frac{9}{10}} = \frac{1}{10^N}.$$

This means that $x = a_1a_2\cdots a_N + 0.00\cdots 01$ where the 1 is in the N th place after the decimal point. So we can write $x = a_1a_2\cdots a_N\dot{9}$ where $a_{N'} = a_N + 1$, e.g. $0.367\dot{9} = 0.368$ and $0.999999\cdots = 0.\dot{9} = 1$.

Generally two distinct decimal expansions that differ in only one place give rise to different numbers. The phenomenon that occurs with repeating nines is a very special one where the notation breaks down and appears to be giving you

two different numbers that are in fact identical. From a common-sense point of view this may be quite obvious as e.g. $\frac{1}{3} = 0.\dot{3}$ and $1 = 3 \times \frac{1}{3} = 3 \times 0.\dot{3} = 0.\dot{9}$.

Just before we conclude this chapter we may ask whether every series $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ converges to give a legitimate decimal expansion. Here the a_n s are chosen from $0, 1, 2, \dots, 9$. It's easy to establish convergence. Since each $a_n \leq 9$ we can use the comparison test as $\sum_{n=1}^{\infty} \frac{9}{10^n} = 1$. Thus we have a complete correspondence between numbers that lie between 0 and 1 and infinite series of the form $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$.

6.12 Exercises for Chapter 6

1. Investigate the convergence of the following series and find the sum whenever this exists

$$(a) \sum_{n=0}^{\infty} \frac{100}{3^n}, \quad (b) \sum_{n=1}^{\infty} n(n+1) \quad (c) \sum_{n=1}^{\infty} \frac{4}{(n+1)(n+2)}$$

$$(d) \sum_{n=1}^{\infty} \frac{1}{n(n+2)}, \quad (e) \sum_{n=0}^{\infty} \sin^n(\theta) \text{ where } 0 \leq \theta \leq 2\pi.$$

2. Use known results about sequences to give thorough proofs that if $\sum_{n=1}^{\infty} a_n$ and

$$\sum_{n=1}^{\infty} b_n \text{ both converge then,}$$

(a) $\sum_{n=1}^{\infty} (a_n + b_n)$ converges.

(b) $\lambda \sum_{n=1}^{\infty} a_n$ converges for all real numbers λ .

Formulate and prove similar results which pertain to divergence in the case where both series are properly divergent to $+\infty$ or to $-\infty$. Why doesn't (a) extend to the case where one of the series is properly divergent to $+\infty$ while the other is properly divergent to $-\infty$?

3. Show that if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} a_n = 0$.

4. (a) Use geometric series to write the recurring decimal $0.\dot{1}\dot{7}$ as a fraction in its lowest terms. [Hint: $0.\dot{1}\dot{7} = \frac{17}{100} + \frac{17}{10000} + \frac{17}{10^6} + \dots$]

(b) Suppose that c is a block of m whole numbers in a recurrent decimal expansion $0.\dot{c}$ (e.g. if $c = 1234$ then $m = 4$ and $0.\dot{c} = 0.\dot{1}23\dot{4}$). Deduce that $0.\dot{c} = \frac{c}{10^m - 1}$.

5. (a) Show that each of the series whose n th term²⁴ is given below diverges

$$(i) (1 + \epsilon)^n \text{ where } \epsilon > 0 \quad (ii) \frac{n+1}{n+2}$$

(b) Find all values of ϵ for which the series whose n th term is $(-1 + \epsilon)^n$

(i) converges, (ii) diverges.

6. Use the comparison test to investigate the convergence of the series whose n th term is as follows:

$$(a) \frac{1}{1+n^2}, \quad (b) \frac{1+\cos(n)}{2^n}, \quad (c) \frac{n}{n^2-1}, \quad (d) \frac{n}{n^3+1}, \quad (e) \frac{2+\sin(n)}{n}.$$

7. Use the ratio test to investigate the convergence of the series whose n th term is as follows:

$$(a) \frac{(n!)^2}{(2n)!}, \quad (b) \frac{2^n}{n!}, \quad (c) \frac{n!}{n^n}, \quad (d) \frac{n^n}{n!}.$$

Note that the solutions to (c) and (d) require some knowledge of the number e , which is discussed in Chapter 7.

8. Use any appropriate technique to investigate the convergence of the series whose n th term is as follows:

$$(a) \frac{1}{n^n}, \quad (b) \sqrt{n+1} - \sqrt{n}, \quad (c) \frac{1}{\sqrt{n(n+1)}}, \quad (d) \frac{n!(n+4)!}{(2n)!}.$$

Once again for (a) it helps if you know about e (or alternatively do Problem 13 first).

9. Show that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative real numbers and (b_n)

is a bounded sequence of nonnegative real numbers, then the series $\sum_{n=1}^{\infty} a_n b_n$ also converges.

10. Show that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers then the

series $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ is also convergent.

²⁴ Here and below by the ' n th term of the series' is meant a_n in $\sum_{n=1}^{\infty} a_n$.

11. Prove the following more powerful form of the ratio test which does not assume that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ exists:

Let (a_n) be a sequence of positive numbers for which there exists $0 \leq r < 1$ and a whole number n_0 such that for all $n > n_0$, $\frac{a_{n+1}}{a_n} \leq r$ then $\sum_{n=1}^{\infty} a_n$ converges.

If $\frac{a_{n+1}}{a_n} > 1$ for all $n \geq n_0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

12. Give thorough proofs of the comparison test and the ratio test for series of the form $\sum_{n=1}^{\infty} a_n$ where (a_n) is an arbitrary real-valued sequence.

13. Although they are the best known, the comparison test and the ratio test are not the only tests for convergent series. Another well-known test is *Cauchy's root test*. This states that if (a_n) is a sequence with each $a_n > 0$ and if there exists $0 < r < 1$ with $\sqrt[n]{a_n} < r$ for all n then $\sum_{n=1}^{\infty} a_n$ converges, but if $\sqrt[n]{a_n} > 1$

for all n then $\sum_{n=1}^{\infty} a_n$ diverges. To prove this result:

(a) Assume $\sqrt[n]{a_n} < r$ with $0 < r < 1$. Show that $a_n \leq r^n$ for all n and hence use the comparison test with a geometric series to show $\sum_{n=1}^{\infty} a_n$ converges.

(b) Suppose that $\sqrt[n]{a_n} > 1$ for all n . Deduce that $\lim_{n \rightarrow \infty} a_n = 0$ cannot hold and hence show that $\sum_{n=1}^{\infty} a_n$ diverges.

(c) Deduce the stronger form of the root test whereby for convergence we only require that $\sqrt[n]{a_n} < r$ for all $n \geq n_0$ where n_0 is a given whole number and for divergence we ask that $\sqrt[n]{a_n} > 1$ for infinitely many n .

14. In Section 6.6 we showed that $\sum_{n=1}^{\infty} \frac{1}{n^r}$ converges whenever $r > 1$. You can use a similar argument to prove another test for convergence of series called *Cauchy's condensation test*: if (a_n) is a nonnegative monotonic decreasing sequence, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Hint: First show that for each natural number k ,

$$a_{2^k} + a_{2^{k+1}} + \dots + a_{2^{k+1}-1} \leq 2^k a_{2^k},$$

$$a_{2^{k+1}} + a_{2^{k+2}} + \dots + a_{2^{k+1}} \geq 2^k a_{2^{k+1}}.$$

15. For each of the following series, decide whether it is (a) convergent, (b) absolutely convergent, giving your reasons in each case.

$$(i) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}, \quad (ii) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n^2 - 2n}, \quad (iii) \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^5}.$$

16. Only one of the following statements is true. Present either a counter-example or a proof in each case.

(a) If $\sum_{n=1}^{\infty} a_n^2$ converges then so does $\sum_{n=1}^{\infty} |a_n|$.

(b) If $\sum_{n=1}^{\infty} |a_n|$ converges then so does $\sum_{n=1}^{\infty} a_n^2$.

17. Give an example of a sequence (a_n) for which $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} a_n^2$ does not.

18. Recall Cauchy's inequality for sums from Exercise 3.9: If a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n are real numbers then

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left(\sum_{i=1}^n a_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n b_i^2 \right)^{\frac{1}{2}}.$$

Now extend this inequality to series, so if (a_n) and (b_n) are sequences and we are given that both $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge, show that $\sum_{i=1}^n a_i b_i$ converges absolutely and that

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left(\sum_{n=1}^{\infty} a_n^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}.$$

19. Suppose that $\sum_{n=1}^{\infty} a_n$ diverges. It is sometimes useful to find the sum of an associated series that really converges. One example of a summation technique for associating a convergent series to a divergent one is *Cèsaro summation*. Recall the sequence of partial sums (s_n) where $s_n = \sum_{i=1}^n a_i$. We define the *Cèsaro average* of the partial sums to be $s'_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$ and if $\lim_{n \rightarrow \infty} s'_n$ exists then we say that the series $\sum_{n=1}^{\infty} a_n$ is *Cèsaro summable*.

- (a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges in the usual sense then it is also Cèsaro summable and that both limits are the same.

- (b) Let $a_n = (-1)^{n+1}$. Show that $\sum_{n=1}^{\infty} a_n$ is Cèsaro summable and that the limit of the Cèsaro averages is $\frac{1}{2}$.