1 Introduction

In this book we are going to generalize theorems about convergence and continuity which are probably familiar to the reader in the case of sequences of real numbers and real-valued functions of one real variable. The kind of result we shall be trying to generalize is the following: if a real-valued function \( f \) is defined and continuous on the closed interval \([a, b]\) in the real line, then \( f \) is bounded on \([a, b]\), i.e. there exists a real number \( K \) such that \(|f(x)| \leq K\) for all \( x \) in \([a, b]\). Several such theorems about real-valued functions of a real variable are true and useful in a more general framework, after suitable minor changes of wording. For example, if we suppose that a real-valued function \( f \) of two real variables is defined and continuous on a rectangle \([a, b] \times [c, d]\), then \( f \) is bounded on this rectangle. Once we have seen that the result generalizes from one to two real variables, it is natural to suspect that it is true for any finite number of real variables, and then to go a step further by asking: how general a situation can the theorem be formulated for, and how generally is it true? These questions lead us first to metric spaces and eventually to topological spaces.

Before going on to study such questions, it is fair to ask: what is the point of generalization? One answer is that it saves time, or at least avoids tedious repetition. If we can show by a single proof that a certain result holds for functions of \( n \) real variables, where \( n \) is any positive integer, this is better than proving it separately for one real variable, two real variables, three real variables, etc. In the same vein, generalization often gives a unified mental grasp of several results which otherwise might just seem vaguely similar, and in addition to the satisfaction involved, this more efficient organization of material helps some people’s understanding. Another gain is that generalization often illuminates the proof of a theorem, because to see how generally a given result can be proved, one has to notice exactly which properties or hypotheses are used at each stage in the proof.

Against this, we should be aware of some dangers in generalization. Most mathematicians would agree that it can be carried to an excessive extent. Just when this stage is reached is a matter of controversy, but the potential reader is warned that some mathematicians would say ‘Enough,
no more (at least as far as analysis is concerned)’ when we get into metric
spaces. Also, there is an initial barrier of unfamiliarity to be overcome in
moving to a more general framework, with its new language; the extent
to which the pay-off is worthwhile is likely to vary from one student to
another.

Our successive generalizations lead to the subject called topology. Ap-
lications of topology range from analysis, geometry, and number theory
to mathematical physics and computer science. Topology is a language for
many mathematical topics, just as mathematics is a language for many
sciences. But it also has attractive results of its own. We have mentioned
that some of these generalize theorems the reader has already met for real-
valued functions of a real variable. Moreover, topology has a geometric
aspect which is familiar in popular expositions as ‘rubber-sheet geo-
metry’, with pictures of doughnuts, Möbius bands, Klein bottles, and the
like; we touch on this in the chapter on quotients, trying to indicate how
such topics are part of the same story as the more analytic aspects. From
the point of view of analysis, topology is the study of continuity, while
from the point of view of geometry, it is the study of those properties
of geometric objects which are preserved when the objects are stretched,
compressed, bent, and otherwise mistreated—everything is legitimate ex-
cept tearing apart and sticking together. This is what gives rise to the old
joke that a topologist is a person who cannot tell the difference between
a coffee cup and a doughnut—the point being that each of these is a solid
object with just one hole through it.

As a consequence of introducing abstractions gradually, the theorem
density in this book is low. The title of theorem is reserved for substantial
results, which have significance in a broad range of mathematics.

Some exercises are marked ⋄ or even ⋄ ⋄ and some passages are en-
closed between ⋄ signs to denote that they are tentatively thought to be
more challenging than the rest. A few paragraphs are enclosed between
▶ and ◀ signs to denote that they require some knowledge of abstract
algebra.

We shall try to illustrate the exposition with suitable diagrams; in
addition readers are urged to draw their own diagrams wherever possible.

A word about the exercises: there are lots. Rather than being daunted,
try a sample at a first reading, some more on revision, and so on. Hints are
given with some of the exercises, and there are further hints on the web
site. When you have done most of the exercises you will have an excellent
understanding of the subject.

A previous course in real analysis is a prerequisite for reading this book.
This means an introduction (including rigorous proofs) to continuity,
differential and preferably also integral calculus for real-valued functions of one real variable, and convergence of real number sequences. This material is included, for example, in Hart (2001) or, in a slightly more sophisticated but very complete way, in Spivak (2006) (names followed by dates in parentheses refer to the bibliography at the end of the book). The experience of abstraction gained from a previous course, in say, linear algebra, would help the reader in a general way to follow the abstraction of metric and topological spaces. However, the student is likely to be the best judge of whether he/she is ready, or wants, to read this book.