Part I
Foundations

What is Discrete Mathematics?
In order to answer that question it is necessary to understand the basic framework used in all areas of Mathematics. This framework will be covered in the first part of the book, Chapters 1–9. After that it will be possible to explain the distinction between *Discrete* Mathematics and other branches of Mathematics.

The key to our answer is to understand what mathematicians do. If, like many readers of this book, you are just starting to study Mathematics at a university or college, you may find parts of the following description rather unexpected.

Mathematicians deal with *statements*.
Usually the statements are about *numbers*.
The statements may be *true* or *false*.
To decide whether a statement is true or false requires a *proof*.

In Chapter 1 we shall introduce the ideas behind the words in italics, by giving some examples. In the subsequent chapters we build on these ideas, by outlining the basic tools used by mathematicians: numbers, sets, and functions.

1
Statements and proofs

1.1 Some mathematical statements

Contents

The following statements are typical of the ones studied by mathematicians. The fact that each one is prefixed with a bullet point (•) is not significant, it is done only for the sake of clarity at this stage.

- 15 *is a multiple of* 3
- 20 *is a multiple of* 3
- 200 000 000 *is a multiple of* 5
- 200 000 000 *is a multiple of* 263
- 200 000 000 *is a perfect square*
- $2^{67} - 1$ *is a prime number*

You will see at once that there are two things to be done. First, we must understand what the statements are about, and then we must decide if they are true or false.

Clearly, the statements are about 'whole numbers' or, as we shall call them, *natural numbers*. These are the objects that we usually denote by the symbols 1, 2, 3, 4 and so on. You are already familiar with their basic properties and, for the time being, we shall take these properties for granted. In Chapter 4 we shall return to this point, and state exactly what we assume about the natural numbers.

The statement • 15 *is a multiple of* 3 is true. We all remember chanting the 'three-times table', in particular the line *five threes are fifteen*. A true statement is known as a **Theorem**, and the reason why it is true is its *Proof*. So we express our conclusion as follows.

Theorem 15 is a multiple of 3.

Proof $15 = 5 \times 3$. □

The square box □ is a useful way of indicating that a proof is finished. Of course, we do not usually make such a fuss about simple results of this kind.

The statement • 20 *is a multiple of* 3 is false. Again, we remember the three-times table, and note that this list of multiples of 3 does not contain 20, although it does contain larger numbers, such as 21. So the statement is not a Theorem. However, a simple twist enables us to express the result formally in the Theorem-Proof style. By inserting the word 'not' we obtain a true statement.

Theorem 20 is not a multiple of 3.

Proof If 20 were a multiple of 3, there would be a natural number n such that $n \times 3 = 20$. Since $7 \times 3 = 21$, n must be less than 7. But none of the values 1, 2, 3, 4, 5, 6 works. □

In due course we shall have a lot more to say about the kind of argument used here. Also, it is worth pointing out that even apparently simple statements, such as the ones we have discussed, are actually rather more complex than they seem.

1.2 How to do mathematics

Many students find difficulty in knowing how to begin answering problems in mathematics. The discussions in this section are intended to help you get started.

In the rest of the book, each section is followed by a set of exercises. The exercises are fairly simple, except in a few cases where a hint is given. This section contains some typical exercises, with a discussion of each one. Remember that when you have worked out how to *answer* a question, you must also consider how to *present* your answer in a clear and concise form. In order to help you do this, there are numerous *Examples* and *Solutions* throughout the book, and many of them contain a model answer, as well as a discussion.

Exercise 1.2.1

Use your knowledge of elementary arithmetic to help you decide whether the following statements are true or false.

- 200 000 000 *is a multiple of* 5
- 200 000 000 *is a multiple of* 263

Discussion All numbers that end in 0 or 5 are multiples of 5. Since we are assuming that the elementary facts of arithmetic are true, we can conclude that the first statement is true.

If asked, you can find the number n such that $200\,000\,000 = n \times 5$: a simple 'division sum' leads to the result $n = 40\,000\,000$.

Unfortunately, there is no way of telling whether a number is a multiple of 263 just by looking at it. Therefore, in order to decide about the second statement you will have to carry out some calculations. The lazy way is to use a calculator to divide 263 into 200 000 000; if the answer is not a whole number, it would suggest that the statement is false. My calculator says that the answer is 760 456.274, and so I conclude that 200 000 000 is not a multiple of 263.

However, when we use a calculator for questions like this, we need to understand its limitations. For instance, the answer 760 456.274 is not exact, it is only an approximation to a number in which the digits go on for ever (whatever that may mean). What if the calculator had said 760 456.001? Is that a natural number with a 'rounding error', or is it really not a natural number?

You can avoid problems like this, and obtain an exact answer, by working out the quotient and remainder when 200 000 000 is divided by 263. You can use your calculator to do this, provided you are strict with it and do not accept any approximations. The simplest way is to use the approximate answer as a guide, and calculate $760\,456 \times 263 = 199\,999\,928$. It follows that

$$200\,000\,000 = 760\,456 \times 263 + 72.$$

Another way of writing this result is

$$\frac{200\,000\,000}{263} = 760\,456 + \frac{72}{263}.$$

Exercise 1.2.2

The numbers 1, 4, 9, 16, 25, 36, 49 *are 'perfect squares'. Write down carefully what you think is meant by the term 'perfect square'.*

Discussion The numbers listed above are 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2. So saying that a natural number n is a perfect square means that we can write it in the form m^2. In other words, n is a perfect square if there is a natural number m such that $n = m^2$.

You might be tempted to answer this question by saying that n is a perfect square if its square root is a natural number. Some of the difficulties raised by this definition are mentioned in the next discussion.

Exercise 1.2.3

My calculator tells me that the square root of 200 000 000 *is* 14142.13562. *What does yours say? What is the right answer? Is the following statement true or false?*

- 200 000 000 *is a perfect square*

Discussion I tried another calculator, and it said that the square root is 14142.13562373. Of course, neither of these is the right answer: they are only approximations (see Chapter 9 for more about this).

Although we may now be rather confused about the meaning of 'square root', we can reach a definite decision about the statement. We can verify for ourselves that $14142^2 = 199\,996\,164$, and $14143^2 = 200\,024\,449$ (note that these are exact answers). Since the squares of 14142 and 14143 are, respectively, smaller and greater than 200 000 000 and there is no natural number between 14142 and 14143, there can be no natural number whose square is 200 000 000.

Exercise 1.2.4

A prime number is a natural number p that cannot be expressed as a product of natural numbers $a \times b$, *except in the trivial ways* $1 \times p$ *and* $p \times 1$. *Using this definition, decide whether the following statement is true or false.*

- 511 *is a prime number*

Discussion If you are not already familiar with the concept of a prime number, you should begin by making a list of prime numbers up to 100:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,
47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

(By convention, 1 is not a prime number.) The list is obtained by writing down all the numbers in order and crossing out those which are *not* primes. You will quickly learn to cross out all the even numbers (multiples of 2), then all multiples of 3, and so on.

The process of listing all the prime numbers goes very slowly, and in order to decide if 511 is a prime you will need a more direct method. The question is: are there natural numbers m and n, neither of them equal to 1, such that $511 = m \times n$? In this case, the question is easily answered by 'trial and error', in other words, by trying small values of m. In fact $511 = 7 \times 73$, so the statement is false.

Exercise 1.2.5

Describe some of the difficulties involved in deciding whether the following statement is true or false.

- $2^{67} - 1$ *is a prime number*

Discussion The number we are asked about, $2^{67} - 1$, is a number with over twenty digits. Even if your calculator can deal with such numbers, you will not be able to carry out all the necessary calculations by the 'trial and error' method. Modern computer algebra systems, such as Maple, contain sophisticated programs for factorizing large numbers, and they will quickly give the factorization:

$$2^{67} - 1 = 193\,707\,721 \times 761\,838\,257\,287.$$

It is remarkable that long before electronic calculators and computers were invented, a mathematician obtained this factorization using only pencil and paper calculations! Of course, it is quite easy to check that the product of the two numbers is $2^{67} - 1$: the difficult part is finding the numbers in the first place.

Exercise 1.2.6

Prove the **Theorem**: $2^{66} - 1$ *is not a prime number.*

Discussion After the previous question you may think that this one is rather unfair. But have faith: there must be a simple way of doing it.

The key lies in simple algebra: $x^2 - 1 = (x - 1)(x + 1)$. Given this hint, you should be able to write out the proof yourself.

1.3 Compound statements

Here are some more statements.

- 81270 *is a multiple of* 2 **and** 5
- 81270 *is a multiple of* 3 **or** 7
- **if** 200 000 000 *is a multiple of* 5, **then** 400 000 000 *is a multiple of* 5
- **if** 200 000 000 *is a multiple of* 375, **then** 400 000 000 *is a multiple of* 375

These are **compound statements**. They are built up from simpler ones using the linking words **and**, **or**, **if - then**. For example, the first statement is a combination of the two statements

- 81270 *is a multiple of* 2
- 81270 *is a multiple of* 5

with the word **and** as the link. In Chapter 3 we shall introduce a slightly more formal notation for this construction, but the meaning is plain as it stands. In order that the combined statement be true, we require that both parts are true. If so, we have a theorem.

Theorem 81270 is a multiple of 2 and 5.

Proof $81270 = 40635 \times 2$, $81270 = 16254 \times 5$. □

The second statement can be written as

- 81270 *is a multiple of* 3 **or** 81270 *is a multiple of* 7

Here we have a problem with the meaning of **or**. Following ordinary usage, the compound statement is true if either of its parts is true. But what if both of them are true? In some circumstances the word 'or' indicates that only one alternative is allowed (*I shall give my best lecture on Monday or Tuesday*) while in others both alternatives are allowed (*I shall give a boring lecture on Monday or Tuesday*). In mathematics we need to settle the meaning once and for all, and we adopt the *inclusive* meaning: a compound **or**-statement is true if at least one of its parts is true.

The third and fourth statements have the form **if** statement-1 **then** statement-2. You should concentrate on the fact that the first word is 'if': we do not say that statement-1 is true, only that *if* statement-1 is true, *then* statement-2 is true.

Example Express the fact that the statement

- **if** 200 000 000 *is a multiple of* 5 **then** 400 000 000 *is a multiple of* 5

is true, in the Theorem-Proof style.

Solution All that is needed is to write down what the words mean and do some very simple algebra. The result should be something like this.

Theorem If 200 000 000 is a multiple of 5, then 400 000 000 is a multiple of 5.

Proof Suppose it is true that 200 000 000 is a multiple of 5. This means that there is a number n such that

$$200\,000\,000 = n \times 5.$$

Now 400 000 000 = 2 × 200 000 000, so it follows that

$$400\,000\,000 = 2 \times 200\,000\,000 = 2 \times (n \times 5).$$

We can rewrite this as $(2n) \times 5$. We have shown that there is a natural number m (in fact $m = 2n$), such that $m \times 5 = 400\,000\,000$; in other words, 400 000 000 is a multiple of 5. □

Note that we did not establish the truth of • 200 000 000 *is a multiple of* 5 (that would have involved giving a definite value for n). We merely showed that if it is true, then • 400 000 000 *is a multiple of* 5 is also true. This point is discussed in Ex.1.3.3.

Exercises 1.3

1 Express the fact that

- 81270 *is a multiple of* 3 **or** 7

is true in the Theorem-Proof style.

2 Express the fact that

- *if* 200 000 000 *is a multiple of* 375 *then* 400 000 000 *is a multiple of* 375

is true in the Theorem-Proof style.

3 Is the statement • 200 000 000 *is a multiple of* 375 true or false? Is this relevant to your answer to the previous exercise?

1.4 Existential statements

An **existential statement** says that something exists. In ordinary language, we use the words **there is** or **there are** to indicate this idea, as in the following statements.

- *there is a number n such that* $n^2 = 441$
- *there are numbers a, b, c such that* $a^2 + b^2 = c^2$
- *there are numbers n and m such that* $n^2 = 2m^2$
- *there are numbers a, b, c, d such that* $a^4 + b^4 + c^4 = d^4$

It is important to be clear about what is meant by ‘a number’ or ‘numbers’ here. For the time being the only numbers we consider are the natural numbers, and we shall say that the natural numbers form the *universe* for our discussion of these statements. In due course, we shall want to use other kinds of numbers, which allow bigger ‘universes’. Then we shall have to specify which universe we are talking about when we make statements about existence.

The first statement is true, because the natural number 21 is such that $21^2 = 441$. Of course, this is how we show that an existential statement is true: we find an *example*. Note that we have only to find one example, and indeed 21 is the only example in this case. (You may also say that $(-21)^2 = 441$, but -21 is not a natural number.)

Finding an example is not always easy. In particular you will be exceedingly lucky if you can find numbers such that $a^4 + b^4 + c^4 = d^4$. Indeed, for hundreds of years many people doubted whether an example existed. During that time the statement was not proved to be either true or false: it was what we call a **Conjecture**. However in 1988, an example was found (see Exercise 1.4.5), so it now has the status of a **Theorem**.

Looking back to Section 1.1, you will see that all of the statements discussed there were, in fact, existential statements. For example, when we said that 15 is a multiple of 3, we meant that *there is* a natural number m such that $15 = m \times 3$.

Exercises 1.4

1 Express the statement • 123 456 789 *is a multiple of* 3 as an existential statement. Is it true?

2 What is the connection between the facts

$$3^2 + 4^2 = 5^2, \quad 5^2 + 12^2 = 13^2, \quad 7^2 + 24^2 = 25^2,$$

and the statement • *there are numbers* a, b, c *such that* $a^2 + b^2 = c^2$?

3 What is the pattern linking the facts given in the previous exercise? [Hint: write down similar equations starting with 9^2, 11^2 and so on.] How would you produce one million examples for the statement?

4 Suppose your friend claims to have an example for the statement

- *there are numbers* n *and* m *such that* $n^2 = 2m^2$.

Let x and y be the natural numbers that (she claims) satisfy the equation $x^2 = 2y^2$. Show that x must be an even number. Deduce that y must be an even number. Can you see why this means that her claim is wrong?

5 One of the following is an example for the statement

- *there are numbers* a, b, c, d *such that* $a^4 + b^4 + c^4 = d^4$

Without doing any complicated arithmetic, decide which one it is. [Hint: look for a simple reason why one of them cannot be an example.]

$$4\,567\,381^4 + 16\,123\,542^4 + 15\,381\,567^4 = 23\,645\,789^4$$
$$2\,682\,440^4 + 15\,365\,639^4 + 18\,796\,760^4 = 20\,615\,673^4$$

1.5 Universal statements

Often we make statements about all numbers, such as the following.

- $n^2 + n$ *is an even number*
- $4n - 1$ *is a prime number*
- *if* n *is a multiple of 6 then* n *is a multiple of 3*
- *if* n *is a multiple of 3 then* n *is a multiple of 6*

These are **universal statements**. Strictly speaking, each of them should be preceded by the phrase *for all natural numbers n*. This phrase makes it clear that, for our present purposes, the universe consists of natural numbers only. If we want to show that a universal statement is true, we have to show that it is true for every one of the natural numbers.

Example Express the fact that the first statement is true in the Theorem-Proof style.

Solution It is not possible to check that $n^2 + n$ is even for all numbers n individually, because the checking would go on for ever. Therefore, we must adopt a different strategy. We check that the result is true if n is even, and also if n is odd. Since these two *cases* cover all possible values of n, the result holds. (In fact there is another approach, leading to an even simpler proof - can you spot it?)

Theorem For all natural numbers n, $n^2 + n$ is an even number.

Proof We have to show that, given n, there is a natural number k such that $n^2 + n = 2k$. We split the proof into two cases.

Case 1: n is even. Let $n = 2r$. Then

$$n^2 + n = 4r^2 + 2r = 2(2r^2 + r),$$

so we can take $k = (2r^2 + r)$ here.

Case 2: n is odd. Let $n = 2s + 1$. Then

$$n^2 + n = (4s^2 + 4s + 1) + (2s + 1) = 2(2s^2 + 3s + 1),$$

so we can take $k = (2s^2 + 3s + 1)$ here.

Since the two cases cover all possibilities, the result is true. □

A universal statement is false if it is false for just one value of n. We refer to a value of n for which the statement is false as a **counter-example**.

Example Write down the ten smallest counter-examples for the statement
• $4n - 1$ *is a prime number*. Is there an unlimited supply of counter-examples?

Solution For $n = 1, 2, 3$ the values of $4n-1$ are 3, 7, 11, all of which are prime numbers, so these are not counter-examples. But 15 is a counter-example, because, when $n = 4$, $4n - 1 = 15$, and 15 is not a prime number. Checking systematically, you should find that the ten smallest counter-examples are:

$$15, 27, 35, 39, 51, 55, 63, 75, 87, 91.$$

In order to find an unlimited supply of counter-examples, you need to spot a pattern. Actually there are two patterns (at least). The most obvious one is 15, 35, 55, 75. These are the numbers 5×3, 5×7, 5×11, 5×15, that is, the numbers of the form $5 \times (4m - 1)$. Now

$$5(4m - 1) = 20m - 5 = 4(5m - 1) - 1.$$

This means that for any n of the form $5m - 1$, the number $4n - 1$ is not a prime, because it can be written as the product $5 \times (4m - 1)$. Thus you have an unlimited supply of counter-examples.

Another pattern starts with the numbers 15, 27, 39, 51, 63, 75, 87. Work this one out for yourself! □

Exercises 1.5

1 Find a counter-example for the statement • $6n + 1$ *is a prime.*
2 Is the statement • *if n is a multiple of 3 then n is a multiple of 6* true or false? If it is true, give a proof, if it is false give a counter-example.
3 State and prove a theorem based on the statement • $n^2 + 7n + 12$ *is not a prime*.
4 State and prove a theorem based on the statement • *if n is a multiple of 6 then n is a multiple of 3*.

1.6 Proof techniques

When we were taught how to do arithmetic, we began with some simple rules, such as the 'times-tables'. Similarly, in order to do mathematics we need to understand some basic proof techniques. We have already come across two of the simplest ones.

In order to show that an existential statement is true: give an example.
In order to show that a universal statement is false: give a counter-example.

Of course, there are many reasons as to why these methods may fail in practice. For example, if an existential statement is actually false, then there is no example. Even if there is an example, we may not be able to find it with the resources available to us (see Exercise 1.4.5).

When we try to find a proof based on logical deductions, the situation may be quite complicated. Sometimes a standard technique, such as splitting the proof into *cases* (Section 1.5), will work. But there is no golden rule that is guaranteed to produce a proof every time.

One very useful technique is known as *proof by contradiction*. We shall discuss its logical basis more in detail in Chapter 3 but, roughly speaking, the idea is as follows.

In order to show that a statement is true, assume the opposite and show that it leads to a false conclusion.

Example Show that 29 is a prime number.

Solution Assume the opposite: suppose 29 is *not* a prime number. Then $29 = rs$ where r and s are not 1 and 29. In fact the smaller of these numbers, say s, must be less than 6, because if both of them are at least 6, we should have $rs \geq 36$. So $s = 2, 3, 4$ or 5. Thus our assumption implies that 29 is a multiple of 2, 3, 4 or 5.

But we know that this is false, because 29 is not one of the numbers in our standard lists of multiples of 2, 3, 4, 5 (the times-tables). Therefore, our assumption must be false, and hence the original statement is true. □

The following theorem, which is very old and very famous, has also been proved by contradiction. We shall discuss the mathematical concepts involved in more detail later, but you already know enough to follow the argument.

Theorem There is no greatest prime number.

Proof Assume the opposite: suppose there is a greatest prime number. Call this number X. Then we can make a complete list of the primes in increasing order: $2, 3, 5, \ldots, X$.

Consider the number

$$Y = (2 \times 3 \times 5 \times \ldots \times X) + 1.$$

Y is greater than X, so it is not a prime. Hence it must have prime factors. But when we divide Y by any one of the primes on our list there is a remainder of 1, so Y is not a multiple of any of these primes. Hence there must be a prime greater than X.

This contradicts our assumption, which must therefore be false, and the original statement is true. □

Exercises 1.6

1 Use the method of proof by contradiction to show that 200 is not a perfect square.

2 Prove that, for all natural numbers n, $n^2 + 7n + 12$ is an even number.

3 Decide whether the following statement is true or false:

• *for all natural numbers* n, $7n + 2$ *is a perfect square.*

If it is true give a proof, if it is false give a counter-example.

4 Explain what is wrong with the following alleged proof that if $n > m$ then $n = m$.

Since $n > m$ we can write $n = m + c$ for some natural number c. Multiplying both sides of this equation by $n - m$ and using elementary algebra we obtain

$$\begin{aligned} n^2 - nm &= nm + nc - m^2 - mc \\ n^2 - nm - nc &= nm - m^2 - mc \\ n(n - m - c) &= m(n - m - c). \end{aligned}$$

Thus $n = m$.

1.7 Miscellaneous Exercises

1 Show that the following statement is false, by giving a counter-example.

- *if* n *is a multiple of* 3 *then* n *is not a multiple of* 7.

2 Show that the following statement is true, by giving a proof.

- *if* n *is a multiple of* 12 *then* n *is a multiple of* 4.

3 Consider the statement

- *given any odd number* a, *there are natural numbers* b, c *such that* $a^2 + b^2 = c^2$.

Show that this is true by using a construction based on the following examples:

$3^2 + 4^2 = 5^2,\ 5^2 + 12^2 = 13^2,\ 7^2 + 24^2 = 25^2,\ 9^2 + 40^2 = 41^2.$

4 Look again at the second Example in Section 1.6. By considering the sequence of numbers 15, 27, 39, 51, 63, 75, 87, construct a proof of the following statement:

- *there is a number* c *such that* $4n - 1$ *is not prime whenever* n *is a number of the form* $cm + 1$.

5 Prove that there are natural numbers x, y such that $x^2 + 4 = y^3$ and $x + y = 16$. How many examples are there?

6 Find an unlimited supply of counter-examples to the statement that $6n + 1$ is a prime.

7 Find an unlimited supply of counter-examples to the statement that $7n + 2$ is a perfect square.