Exercise 7.11.1

Take $A \subseteq B$. If define $\bar{b} = \max B$.If $\bar{b} \notin A$ then $a < \bar{b}$ for every $a \in A$ since $a \in A$ implies $a \in B$ and $\bar{b}$ is the maximal element in B. If $\bar{b} \in A$ then $\max A = \bar{b}$ since as before all the other elements of A are smaller.

Exercise 7.11.2

Let $a_l = \max\{a_1, ..., a_n\}$ and $b_k = \max\{b_1, ..., b_n\}$. If $l = k$, then $a_l + b_l \geq a_i + b_i$ for all $i \neq l$, and we get the equality. But if $l \neq k$, then $a_l + b_k > \max\{a_1 + b_1, ..., a_n + b_n\}$.

Exercise 7.11.3

Let $\bar{a} = \max\{-a_1, -a_{2,}..., -a_n\}$ and $\underline{a} = -\min\{a_1, a_{2,}..., a_n\}$ and assume first that all a_i were positive. Then $\bar{a}$ is the least negative element among $\{-a_1, -a_{2,}..., -a_n\}$. But that is precisely the negative of the $\min\{a_1, a_{2,}..., a_n\}$. If some of the a_i are negative then the argument is exactly the same, interpreting "least negative" as "most positive". Similarly for the other equality.

Exercise 7.11.4

maximin$A$ = 3 =minimax$A$; maximin$B$ = 2, minimax$B$ = 3, maximin$C$ = 2, minimax$C$ = 3, maximin$D$ = 1 =minimax$D$.

Exercise 7.11.5

This follows from the repeated application of exercise 3 to the rows or columns of A. Denote by A_i the i-th row and by A^j the j-th column of a matrix A. Then $-A_i = \left(-A^T\right)^i$ so

$$\begin{aligned} maximin\left(-A^T\right) &= \max\left\{\min\left\{-A_1\right\}, \min\left\{-A_2\right\}, ..., \min\left\{-A_n\right\}\right\} \\ &= \max\left\{-\max\left\{A_1\right\}, -\max\left\{A_2\right\}, ... -\max\left\{A_n\right\}\right\} \\ &= -\min\left\{\max\left\{A_1\right\}, \max\left\{A_2\right\}, ... \max\left\{A_n\right\}\right\} \\ &= -\text{minimax}\left(A\right). \end{aligned}$$

Exercise 7.11.6

The saddle points are a_1^2 for A and d_1^4, d_2^4 and d_3^4 for D. The rest of the matrices don't have saddle points.

Exercise 7.11.7

For any matrix A, $\max_{s\in S}\min_{t\in T}\pi\left(s,t\right)=$maximin$A$, and $\min_{s\in S}\max_{t\in T}\pi\left(s,t\right)=$minimiax$A$. The answer follows from exercise 4.

Exercise 7.11.8

An $m \times 1$ matrix only has one column, which has a unique maximal element. Since this element is the only element in the row, it is a saddle point.

Exercise 7.11.9

sup = 2, inf = 1. For every $x < 2$ there exists a y satisfying $x < y < 2$, so the only candidate for a maximum is 2. However, $2 \notin (1, 2)$. Similarly for the min.

Exercise 7.11.10

Player one's security level in A is 3, security strategy is A_2 (player II should play A^1), in D the security level is 1, playing whichever of the three strategies (II should play D^4).

Exercise 7.11.11

For A: II's security starategy is $\left(A^T\right)^1$, security level -3, I should play $\left(A^T\right)_2$. For D : II's security strategy is $\left(D^T\right)^4$, security level -1, I can play anything.

Exercise 7.11.12

By definition $\overline{m} = \min \sup \pi(d,e)$. Since $\pi(d,e) = \begin{cases} p_1(d), \text{ if } d > e \\ 1 - p_2(e) \text{ if } d \leq e \end{cases}$ we get that $m(e) = \sup_d \pi(d,e) = \begin{cases} p_1(e) \text{ if } p_1(e) > 1 - p_2(e) \\ 1 - p_2(e); \text{ else} \end{cases}$. Thus, $\min_e m(e) = p_1(\delta) = 1 - p_2(\delta)$ is the saddle point.

Exercise 7.11.13

First note that all the columns of the payoff matrix lie on the line $y = 11 - 2x$. Thus equating $x = 11 - 2x$ we find that the security level for player I is $v = \frac{11}{3}$. The normal vector to that line $y = 11 - 2x$ is $p = (2, 1)$., so I's security strategy is $\left(\frac{2}{3}, \frac{1}{3}\right)$.

Exercise 7.11.14

Since any strategy is a security strategy for player I, any mix of his strategies is also a security strategy.

Exercise 7.11.15

If I uses $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$, then whichever strategy II uses, I always obtains an expected payoff of at least 3. However, if II uses a strategy other than his pure strategy C^4, I gets more than 3.

Exercise 7.11.16

Observe that the rows of the matrix define the line $x + y = 5$. Thus $v = 2.5$ and $p = \left(\frac{1}{2}, \frac{1}{2}\right)^T$.

Exercise 7.11.17

$f(x,y) = (1-x)(1-y) + 3(1-x)y + 4x(1-y) + 2xy = 1 + 3x + 2y - 4yx$ and $\frac{\partial f}{\partial x} = 3 - 4y$ and.$\frac{\partial f}{\partial y} = 2 - 4x$. So the point $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ is the point $x = \frac{3}{4}$, $y = \frac{1}{2}$. To see that this is a saddle point notice that when $y < \frac{1}{2}$ $f(x,y)$ is an increasing function of x, but when $y > \frac{1}{2}$ $f(x,y)$ is a decreasing function of x. Similarly for the other coordinate. Thus, $\left(\frac{3}{4}, \frac{1}{2}\right)$ cannot be either a maximum or a minimum, hence it has to be a saddle point.[1]

[1]Alternatively, one can compute the matrix of the second derivatives and see that it is neither positive nor negative definite at $\left(\frac{3}{4}, \frac{1}{2}\right)$.

Exercise 7.11.18

Denote I's payoff in a Nash equilibrium by v. best replies to II's strategy. If v were less than the minimax, then I could deviate to playing the security strategy and obtaining strictly more, contradicting the fact that in a Nash equilibrium strategies are best replies.

Exercise 7.11.19

In 6.15(b) observe that when *Adam* plays the mixed strategy $\left(\frac{1}{3}, \frac{2}{3}\right)$ he gets the same payoff regardless of what strategy *Eve* uses. Thus, this is Adam's security strategy. Similarly, *Eve*'s security strategy is $\left(\frac{2}{3}, \frac{1}{3}\right)$. A mixed equilibrium is given by strategies $\left(\frac{2}{3}, \frac{1}{3}\right)$ for *Adam* and $\left(\frac{1}{3}, \frac{2}{3}\right)$ for *Eve*. By symmetry, the payoffs from playing the mixed equilibria are the same as the payoffs from playing security strategies - both are equal to $\frac{2}{3}$. Mixed equilibrium strategies are not secure because if *Eve* plays *ball* then if *Adam* plays the equilibrium strategy, he obtains only an expected payoff $\frac{1}{3}$, which is less than his security payoff. Similarly, in 6.15(a) the mixed strategies are given by $\left(\frac{1}{2}, \frac{1}{2}\right)$ for both players. These are also the security strategies of the players, yielding the expected payoffs of 1 to each player.

Exercise 7.11.20

(a) Let p be Adam and Eve's common probability that a Republican will win. Then Eve's expected payoff is $10p - 10(1-p)$ and Adam's expected payoff is $10(1-p) - 10p$. Summing them we get zero.

(b) In this situation Eve's expected payoff is $10\frac{3}{4} - 10\frac{1}{4} = 5$. Adam's expected payoff is $10\frac{5}{8} - 10\frac{3}{8} = 2.5$. Summing we get 7.5; hence, this is not a zero-sum game.

(c) Again, let p be Adam and Eve's common probability that a Republican will win. Then Eve's expected payoff is $u_E(10)p + U_E(-10)(1-p) < u_E(0)$, and Adam's expected payoff is $u_A(-10)p + u_A(10)(1-p) < u_A(0)$. These inequalities arise because both are strictly risk-averse and hence have strictly concave utility functions. Since both players have an expected utility less than that with which they started, the game cannot be zero-sum.

Exercise 7.11.21

In a game with a payoff matrix $-A^T$, player II has the same strategy choices and equivalent payoffs as player I i a game with A as the payoff matrix. A matrix game with a skew-symmetric payoff matrix is called "symmetric" because both players are indifferent to whether they are player I or II. Since both players have the same strategy choices and would be indifferent to whether they are player I or player II, the value of such a game must be zero. If it were not zero, the players would not be indifferent as to which player they were. More formally, let v equal the value to player I and $-v$ be the value to player II. Then by theorem 7.9 we know $v = \underline{v} = \overline{v}$, and player I can ensure that he gets an expected payoff of v or more by using one of his security strategies $\tilde{p}$. Player II can also ensure she gets $-v$ or more by playing one of her security strategies $\tilde{q}$. However, since the game is symmetric it must be true that $v = -v$, and so $v = 0$.

Exercise 7.11.22

(a) Security strategies

$$\tilde{p} = \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix} \quad \tilde{q} = \begin{bmatrix} 0 \\ \frac{7}{16} \\ 0 \\ \frac{9}{16} \\ 0 \end{bmatrix}.$$

(b) Security strategies

$$\tilde{p} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad \tilde{q} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}.$$

The above are also the Nash equilibrium strategies.

Exercise 7.11.23

(a)

$$v = 1 \quad \tilde{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \tilde{q} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(b)

$$v = 1 \quad \tilde{p} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad \tilde{q} = \begin{bmatrix} a \\ b \\ a \end{bmatrix}.$$

(c)

$$c = -2 \quad \tilde{p} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \quad \tilde{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Exercise 7.11.24

(a) The security strategies are $\tilde{p} = s_3$ and $\tilde{q} = t_3$; $v = 3$

(b)

$$v = \frac{18}{7} \quad \tilde{p} = \begin{bmatrix} 0 \\ \frac{1}{7} \\ \frac{6}{7} \\ 0 \end{bmatrix} \quad \tilde{q} = \begin{bmatrix} \frac{4}{7} \\ 0 \\ \frac{3}{7} \\ 0 \end{bmatrix}.$$

Exercise 7.11.25

(a) Consider the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Using the method of section 7.4.3, we know that a security strategy $(1 - r, r)$ must satisfy $a(1 - r) + cr = b(1 - r) + dr = v$. This just says that player I's payoff must be the same whatever player II does.

(b) Since player I's payoff is the same whatever player II does, player II must get the same whatever she does because the game is zero-sum.

Exercise 7.11.26

Use the matrix A of exercise 7.11.25. Simple calculations for its mixed equilibrium give $v = \frac{bc-ad}{c-a-d+b}$. Calculating $(e^T A^{-1} e)^{-1}$ yields the same result.

Exercise 7.11.27

In order to find the shadow prices y, Alice solves the dual problem to her maximization problem, i.e. $\min y^T b$ s.t. $y^T A \geq c^T$ and $y \geq 0$. An easy way to find the solution to this is to proceed geometrically. First, the set $\left\{y | y^T A \geq c^T \text{ and } y \geq 0\right\}$ is convex as it is an intersection of convex sets (half-planes), and its boundary is made out of straight line segments. Next, the objective function $y^T b$ takes constant values on straight lines defined by $y^T b = v$, where v is some constant. Thus, we need to find the smallest v such that $\left\{y | y^T b = v\right\} \cap \left\{y | y^T A \geq c^T \text{ and } y \geq 0\right\} \neq \emptyset$. Geometrically, this is easy since $\left\{y | y^T A \geq c^T \text{ and } y \geq 0\right\} = \{y | y_1 + 4y_2 \geq 1\} \cap \{y | 3y_1 + 2y_2 \geq 1\}$, so one boundary of the feasible set is actually parallel to the level lines of the objective function. Thus, in order to minimize $y^T b$, we just have to move it all the way to that boundary, that is $y^T b = 1$. From $y_1 + 4y_2 \geq 1$ we then obtain that $y_1 \in \left[0, \frac{1}{5}\right]$ and $y_2 = \frac{1}{2}(1 - 3y_1)$.

Exercise 7.11.28

The only way that the solution to the dual problem is unique is when the level lines of the objective function $y^T b$ aren't parallel to any of the boundaries of the feasible set $y^T A \geq c^T$. But then the minimization of $y^T b$ happens at a corner of the boundary (e.g. when at least two constraints are met with equality). Changing b changes the slope of the level lines $y^T b = v$, and since these weren't parallel to the boundary with stock b, then they also won't be parallel for a small enough change of b, assuring that the minimization occurs at the same corner of the feasible set as before. In order for Alice to want to buy small amounts of raw materials it has to be that $p \neq y$.

Exercise 7.11.29

(a) $v = 2$.

(b) $v = 1$.

(c) $v = \frac{3}{2}$.

Exercise 7.11.30

The strategic form of this game is

	(2,1)	(1,2)
(3,1)	2	0
(2,2)	1	1
(1,3)	0	2

.

Inspection reveals that the above matrix has no saddle point. A mixed strategy equilibrium has Colonel Blotto choosing $(3,1)$, $(2,2)$, and $(1,3)$ with probability $\frac{1}{3}$ each. Count Baloney will choose $(2,1)$ and $(1,2)$ with probability $\frac{1}{2}$ each.

Exercise 7.11.31

The strategic form of this game is

	(3,1)	(1,3)	(2,2)
(4,1)	3	0	1
(1,4)	0	3	1
(3,2)	1	-1	2
(2,3)	-1	1	2

,

and the solution is illustrated in the following table:

	$\{(3,1), (1,3)\}$	(2,2)
$\{(1,4), (4,1)\}$	$\frac{3}{2}$	1
$\{(3,2), (2,3)\}$	0	2

.

Inspection reveals that the matrix has no saddle point. A mixed strategy equilibrium has Colonel Blotto choosing $(3,2)$ or $(2,3)$ with probability $\frac{1}{5}$ and $(1,4)$ or $(4,1)$ with probability $\frac{4}{5}$. Count Baloney will choose $(3,1)$ or $(1,3)$ with probability $\frac{2}{5}$ and $(2,2)$ with probability $\frac{3}{5}$.

Exercise 7.11.32

Eve will always win this game. Her winning strategy is to attack one of the two center spaces with her first bomb. If it is a hit, she can bomb both sides with her next two turns. If it is a miss, she knows the battleship must be in the only two adjacent spaces remaining. She hits these with her second and third bombs.

Exercise 7.11.33

On day 1, the n-Inspection Game either ends or (which is the case if either one chooses to act), otherwise the players enter the $(n-1)$-Inspection Game on day 2, hence figure 7.19(a). From the figure, we obtain the desired difference equation by computing the security strategy of player I. Let p be the probability with which he plays *act*, then I's payoffs given II's actions are $E_{act}(p) = 1 - 2p$, and $E_{wait}(p) = v_{n-1} + p(1 - v_{n-1})$. I's security strategy is given by p which solves $E_{act}(p) = E_{wait}(p)$, hence, $p = \frac{1-v_{n-1}}{3-v_{n-1}}$. From here, we can compute the value of the game, which is $v_n = \frac{1+v_{n-1}}{3-v_{n-1}}$. Substituting $v_n = 1 - w_n^{-1}$ we obtain the difference equation for w_n, $w_n = w_{n-1} + \frac{1}{2}$, so that $w_n = \frac{n}{2} + w_0$. Plugging this into v_n we get $v_n = 1 - \frac{2}{2w_0+n}$, and using the boundary condition, $v_1 = -1$, we get that $w_0 = 0$ and $v_n = 1 - \frac{2}{n}$.

Exercise 7.11.34

Proceed as in exercise 7.11.33, but with figure 7.19 modified so that its top-right cell is v_{n-1} and its bottom-right cell is u_{n-1}. Then $E_1(r) = 1 - 2r$ and $E_2(r) = (1-r)v_{n-1} + ru_{n-1}$. Player I's security strategy $\tilde{r}$ is found by setting these equal. The formula for u_n then follows by writing $u_n = E_1(\tilde{r})$. Since $u_2 = 1$, it follows that $u_3 = \frac{1}{3}$ and therefore $u_4 = 0$. The agency should inspect with probability $\frac{1}{2}$ the first day.

Exercise 7.11.35

The optimal strategies are for Colonel Blotto to send 1 or 2 companies each with probability $\frac{1}{2}$. Count Baloney sends 0 or 1 companies each with probability $\frac{1}{2}$. Colonel Blotto's expected payoff is v_n, where $v_n = \frac{1}{2}(1 + v_{n-1})$. Since $v_0 = 0$, it follows that $v_n = 1 - (\frac{1}{2})^n$.

Exercise 7.11.36

The only security strategy is to choose *heads* and *tails* with equal probability. It is a Nash equilibrium if all players choose *heads*. For a two-player zero-sum game, a strategy profile is a Nash equilibrium if and only if each player uses one of his or her security strategies.

Exercise 7.11.37

(a) $v = 0.6713$, $p = \begin{bmatrix} 0.3636 \\ 0.3497 \\ 0.2867 \end{bmatrix}$.

(b) $v = 3.1429$, $p = \begin{bmatrix} 0.5714 \\ 0.4286 \\ 0 \end{bmatrix}$.