Exercise 6.9.1

$p = \left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right)$. To implement this strategy a player could throw a die and play the first strategy if the result were 1 or 2, the third strategy if the result were 3 or 4 and the fourth one otherwise.

Exercise 6.9.2

Let H denote the strategy "Help" and NH the strategy "Refuse to Help". Assume $N \geq 2$ and that players $n = 2, ..., N$ play H with probability $(1-p)$ and NH with probability p. Thus the payoff to player 1 if he plays H is $1-c$, and $p^{N-1}\,0 + \left(1-p^{N-1}\right)1 = \left(1-p^{N-1}\right)$ if he plays NH. In order for 1 to play a mixed strategy it must be that$1 - p^{N-1} = 1 - c$, thus $p = c^{\frac{1}{N-1}}$. As $N \to \infty$, this goes to 1 (it is not difficult to verify that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for any $a > 0$). Since the probability that the man is helped is $1 - p^N = 1 - c^{\frac{N}{N-1}}$, it follows that as $N \to \infty$, this probability goes to $1 - c$.

Exercise 6.9.3

Let p the probability that a punter chooses b and $(1-p)$ be the probability that he chooses c. In a mixed strategy equilibrium it musy be that a punter is indifferent between playing either of the two strategies. Given the strategies of the other two punters, playing b yields the payoff $\frac{1}{3}\left[\frac{1}{12}p^2 - \frac{1}{3}p + \frac{1}{2}\right]$ and playing c yields $\frac{1}{3}\left[\frac{1}{3} + \frac{1}{3}p + \frac{1}{3}p^2\right]$. Equating these two expressions gives the equation for p, $3p^2 + 8p - 2 = 0$, and the resulting value for p is approximately $p = 0.23$.

Exercise 6.9.4

See figure 6.9.4. There is no pure strategy equilibrium because the reaction curves do not cross.

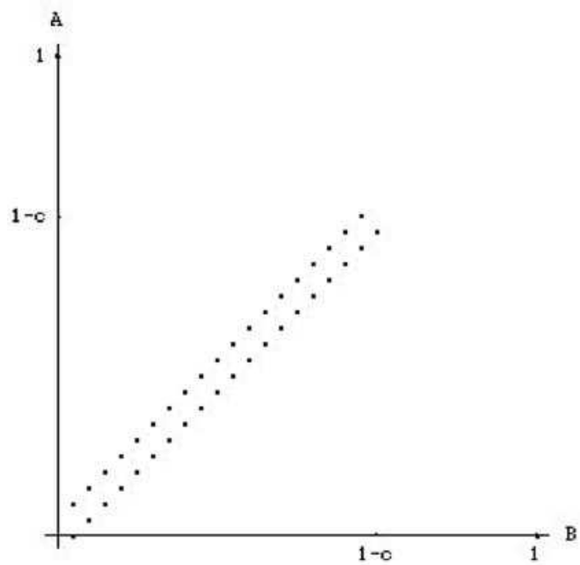


Figure 6.9.4

Exercise 6.9.5

It is strongly dominated by staying out and getting 0 (as opposed to entering and getting something negative).

Exercise 6.9.6

Suppose A bids m with a probability p. Then it is optimal for B to put probability 0 on some interval of bids below m. Hence, for A, bidding m is dominated by bidding $m - \varepsilon$, for some $\varepsilon > 0$.

Exercise 6.9.7

Now it is easy to analyse the game using backward induction. In the subgame where Alice enters and Bob doesn't, Alice gets $1-c$ and Bob gets 0; similarly for the reverse case. In the subgame where both enter, there is a unique pure strategy equilibrium in which both bid 1, and each obtains a final payoff of $-c$. Why? Suppose Alice bids some number $a < 1$, then if Bob bids $b < a$ he makes a loss of $-c$ (the entry fee), while if he bids $a < \tilde{b} < 1$ he makes $1 - \tilde{b} - c > c$. Thus the only possibility is for both to bid 1. If none enters, then clearly each obtains a payoff of 0. Thus, backward inducting, when Alice and Bob are deciding whether to enter or not, they are facing the following payoff matrix:

	enter	out
enter	$-c$ $-c$	0 $1-c$
out	$1-c$ 0	0 0

This game has a unique mixed strategy equilibrium where each enters with probability $p = 1 - c$.

Exercise 6.9.8

Let the bimatrix game be given by

	p_2	$1-p_2$
p_1	a_1, a_2	b_1, b_2
$1-p_1$	c_1, c_2	d_1, d_2

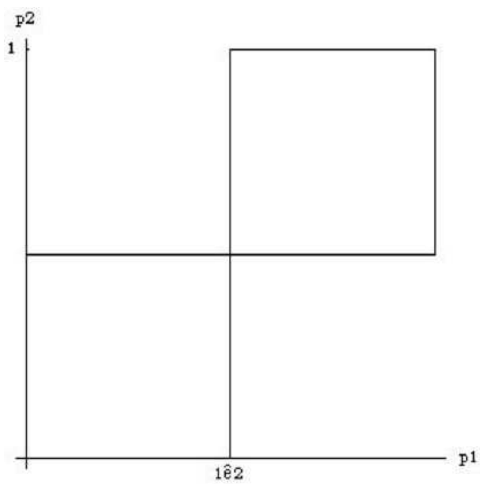
The reaction curve of I is given by

$$r(p_2) = \begin{cases} U \text{ if } p_2 a_1 + (1-p_2)\, b_1 > p_2 c_1 + (1-p_2)\, d_1 \\ U \text{ with } p_1 \text{ and } D \text{ with } 1-p_1 \text{ if there is equality, } \ p_1 \in [0,1] \\ D \text{ if } p_2 a_1 + (1-p_2)\, b_1 < p_2 c_1 + (1-p_2)\, d_1 \end{cases}$$

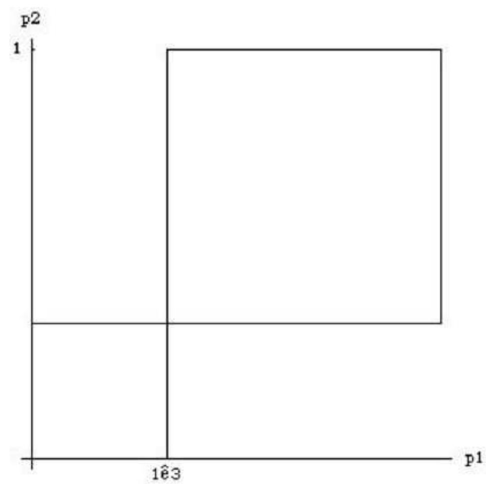
If γ is added to I's payoffs in column 1, then playing U gives $p_2(a_1+\gamma)+(1-p_2)\,b_1$ and D yields $p_2(c_1+\gamma)+(1-p_2)\,d_1$. Since $p_2(a_1+\gamma)+(1-p_2)\,b_1 > p_2(c_1+\gamma)+(1-p_2)\,d_1$ if and only if $p_2a_1+(1-p_2)\,b_1 > p_2c_1+(1-p_2)\,d_1$, it follows that I's reaction curve does not change.

Exercise 6.9.9

For Chicken see figure 6.9.9(a), for BoS see figure 6.9.9(b). We can see that all the Nash equilibria of Chicken are $(Slow, Speed), (Speed, Slow)$ and a mixed equilibrium $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. The Nash equilibria for the BoS $(Box, Ball), (Ball, Box)$ and $\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$.



(a) Figure 6.9.9(a)



(b) Figure 6.9.9(b)

Exercise 6.9.10

If a player who is to play *dove* plays *hawk* instead, then his payoff is $\frac{1}{N}\left[\left(\frac{N}{3}-1\right)4-\frac{2}{3}N\right]=\frac{2}{3}-\frac{4}{N}$. If he played *dove*, then he would obtain a payoff of $\frac{2}{3}-\frac{1}{N}$. On the other hand, if a player who was supposed to play *hawk*, thus obtaining a payoff of $\frac{1}{N}\left[\frac{N}{3}4-\left(\frac{2}{3}N-1\right)\right]=\frac{2}{3}+\frac{1}{N}$ had switched to *dove*, he would obtain a payoff of $\frac{2}{3}$. Hence, we need $\varepsilon>\frac{3}{N}$.

Exercise 6.9.11

(a) No, because A and B have different dimensions.

(b) Yes.

$$B + C = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 4 \end{bmatrix}$$

(c) No.

(d) Yes.

$$3A = \begin{bmatrix} 6 & 3 & 9 \\ -3 & 12 & 0 \end{bmatrix}$$

(e) Yes.

$$3B - 2C = \begin{bmatrix} 3 & 4 \\ -2 & -1 \\ 9 & -8 \end{bmatrix}$$

(f) Yes.

$$A - (B + C)^T = \begin{bmatrix} 1 & 2 & 0 \\ -4 & 3 & -4 \end{bmatrix}$$

Exercise 6.9.12

(a) AB is meaningful because A has 2 columns and B has 2 rows; BA doesn't go because B has 2 columns and A has 3 rows.

$$AB = \begin{bmatrix} 4 & 0 \\ 2 & 4 \\ 6 & 0 \end{bmatrix}$$

(b) Because both are square matrices; $BC \neq CB$.

Exercise 6.9.13

Multiply matrix A with the vector x and write the equality with b to obtain the system of linear equations.

Exercise 6.9.14

(a)

$$x + y = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

(b)

$$x + y = \begin{bmatrix} 12 \\ -9 \end{bmatrix}$$

(c)

$$-2z = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

(d)

$$-2z = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

(e)

$$2x + y = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

Exercise 6.9.15

$x^T y$ is a number and xy^T is a $n \times n$ matrix; $x^T y$ and $y^T x$ are both the same numbers.

Exercise 6.9.16

(a) 14

(b) -9

(c) -1

(d) 0

(e) $\sqrt{14}$

(f) $\sqrt{46}$

Exercise 6.9.17

(a) $\sqrt{14}$

(b) $\sqrt{46}$

(c) Vectors y and z.

Exercise 6.9.18

In matrix C. The dominated strategy is the 4-th column, and the dominating mixed strategy puts probability $\frac{1}{3}$ on each of the columns 1,2, and 3.

Exercise 6.9.19

$Bq \leq \beta e$, $p^T B = \alpha e$.

Exercise 6.9.20

$p^T B = \alpha e$. Thus, $3p_2 = \alpha$ and $p_1 + p_2 = \alpha$, and taking into account $p_1 + 3p_2 = 1$ yields $p_1 = 0.4$ and $p_2 = 0.2$.

Exercise 6.9.21

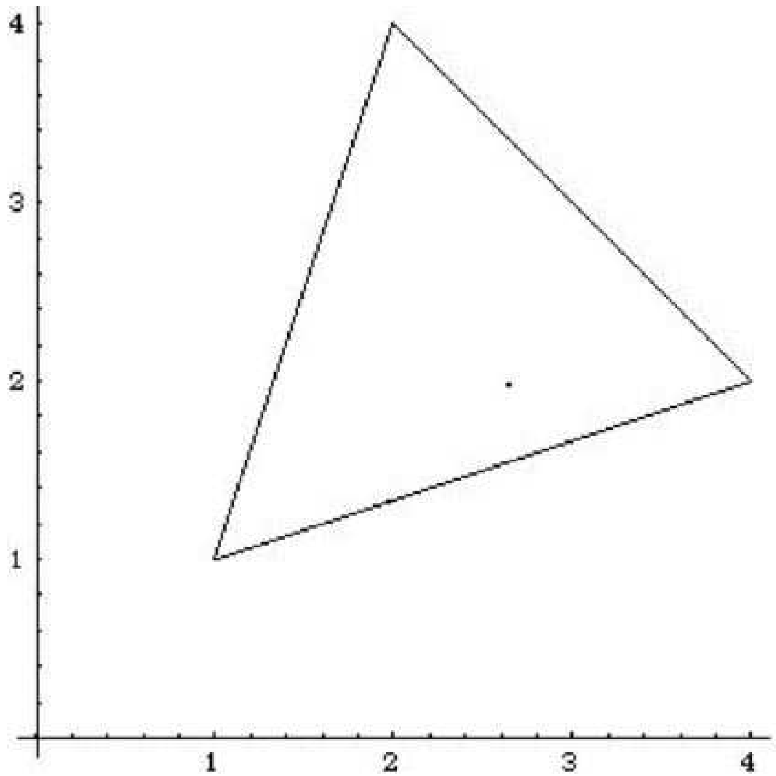
The first inequality means that there is a mixed strategy q of player II, such that any mixed strategy of player one yields to her at most as much as $\tilde{p}$ does. Thus, $\tilde{p}$ is a best reply. The second inequality means that there is a strategy p which yields a payoff to I that is strictly greater than her payoff from $\tilde{p}$ for any strategy of II, which is precisely the definition of $\tilde{p}$ being strongly dominated by p. Similarly for the third inequality.

Exercise 6.9.22

Notice that $(3-2\alpha, 2, 1+2\alpha) = \alpha\,(1,2,3) + (1-\alpha)\,(3,2,1)$, which describes all the points on the straight line from $(1,2,3)$ to $(3,2,1)$, for $\alpha \in [0,1]$. For $\alpha = \frac{1}{2}$ we obtain the mid-point, and for $\alpha = \frac{2}{3}$ we obtain the center of gravity for mass $\frac{1}{3}$ at x and mass $\frac{2}{3}$ at y.

Exercise 6.9.23

$(3,3) = \frac{1}{2}(4,2) + \frac{1}{2}(2,4)$. See figure 6.9.23.



Exercise 6.9.24

(a) A is a circumference with radius 2 and centered at $(0, 0)$. Not convex.

(b) B is the circle bounded by A, and it is convex.

(c) C is line, parallel to x_2-axis, it is convex.

(d) D is a half-plane, to the left of C, it is convex.

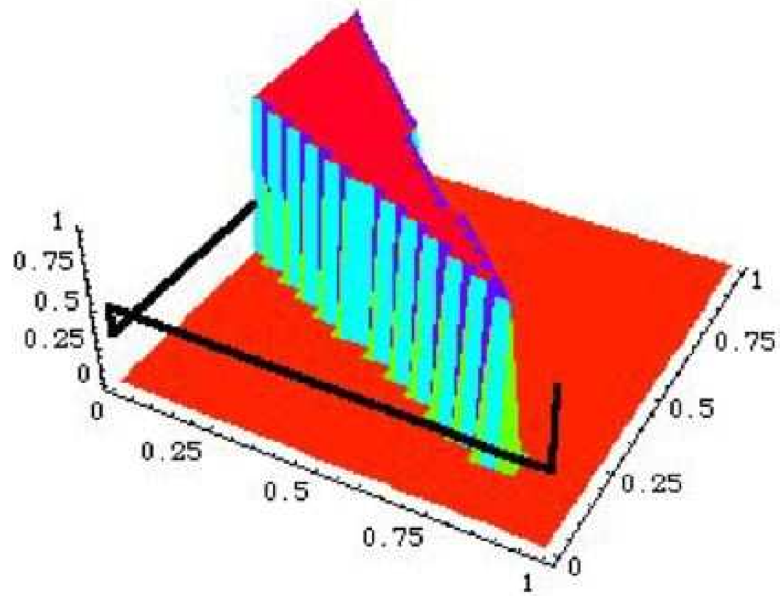
(e) E is two perpendicular lines, intersecting at the point $(4, 4)$, not convex.

Exercise 6.9.25

The set $\{(1-\gamma)\,u+\gamma z, \gamma\in[0,1]\}$ is a straight line connecting u and z. If we take $\gamma=0$, we obtain v, thus $v=(1-\gamma)\,u+\gamma z$ is the point γ away from u. For the second part notice that the equations $w=\alpha x+\beta y+\gamma z$, $\gamma=\pi_3$, $\alpha+\beta+\gamma=1$, describe the set of points $\{\alpha\,[(1-\pi_3)\,x+\pi_3\,(z-y)]+(1-\alpha)\,[y+\pi_3(z-y)]\,,\alpha\in[0,1]\}$, which is a straight line connecting the points $y+\pi_3(z-y)$ and $(1-\pi_3)\,x+\pi_3\,(z-y)$.

Exercise 6.9.26

See figure 6.9.26.

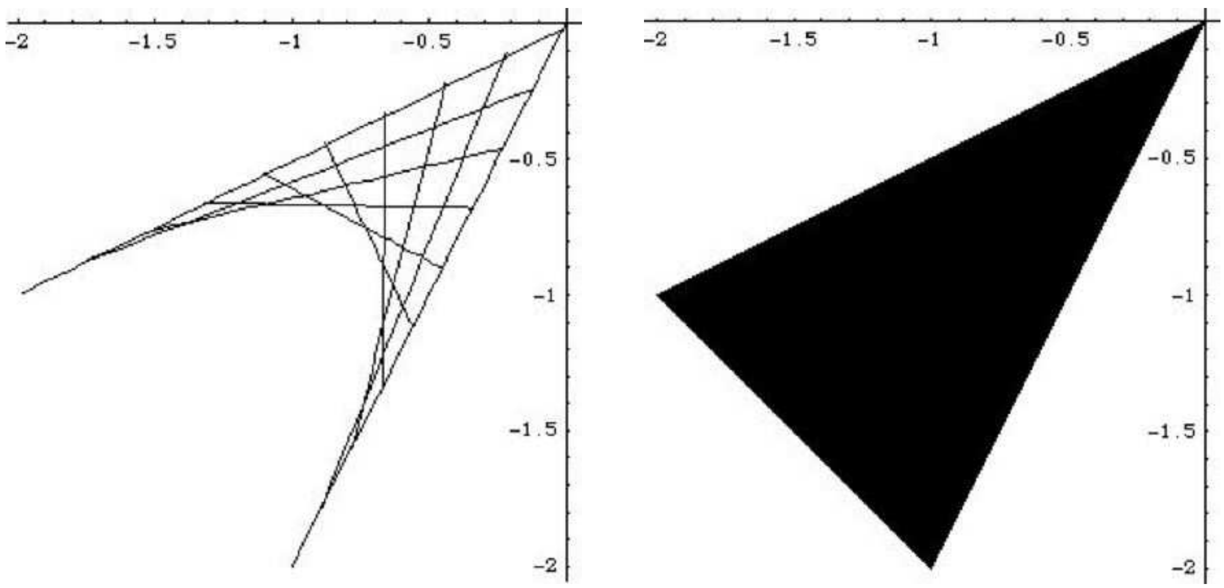


Exercise 6.9.27

The function is affine iff $f(\alpha x_1 + \beta \tilde{x}_1, \alpha x_2 + \beta \tilde{x}_2) = \alpha f(x_1, x_2) + \beta f(\tilde{x}_1, \tilde{x}_2)$ for any α, β and any $x_1, x_2, \tilde{x}_1, \tilde{x}_2$. It is immediate to check that this is true for our f. The points are $f(1,1) = (4,5)$, $f(2,4) = (11,10)$, $f(4,2) = (9,12)$.

Exercise 6.9.28

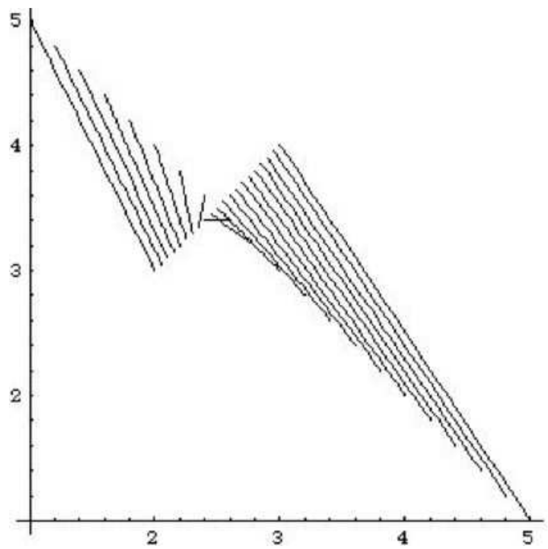
For the non-cooperative region see figure 6.9.28(a), and for the cooperative region see figure 6.9.28(b). The convex hull of either one of them is the cooperative region.



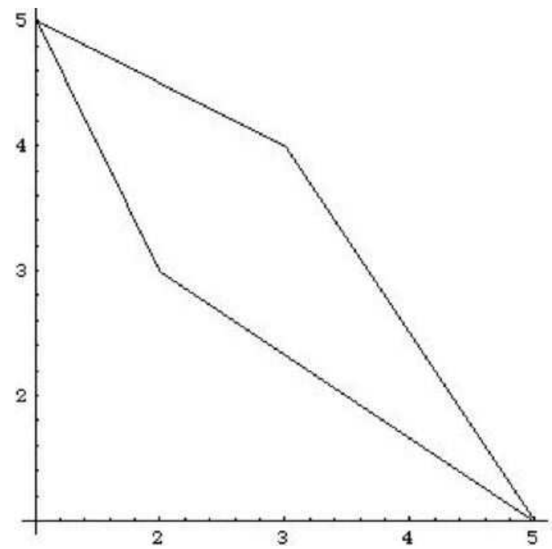
(a) Figure 6.9.28(a) (b) Figure 6.9.28(b)

Exercise 6.9.29

See figures 6.9.29(a) and 6.9.29(b).



(a) Figure 6.9.29(a)



(b) Figure 6.9.29(b)

Exercise 6.9.30

To find the set of correlated equilibria, we will use the fact that it is a convex set, bounded by straight lines. To see this, notice that the correlated equilibrium conditions will give linear inequalities for the probabilities, hence the set of probabilities of different outcomes will be convex. Since expected payoffs are linear combinations of those probabilities, it must be that the set of possible payoffs is convex. Thus, it is enough to find its extreme points (corners). Let the strategies of the two players and the probabilities with which different outcomes are played be denoted as in figure 6.19(b). First notice that the pure Nash equilibria are in the set of the correlated equilibria (set $e = 1$ and it is by the definition of Nash equilibrium optimal for each player to play her equilibrium strategy). These have to be two extreme points of the set, since none of the two players could ever get more than 3. To find the other two corners notice that since the game is symmetric, they will correspond to the cases when $f = g$. First assume $h = 0$ so that $e = 1 - 2g$, $prob\,(A|C) = \frac{e}{e+f} = \frac{1-2g}{1-g}$, $prob\,(B|C) = \frac{g}{1-g}$, $prob\,(A|D) = 1$, and $prob\,(B|D) = 0$. So the expected payoff of II when she is instructed to play C and does so is $U\,(C|C) = 2\frac{1-2g}{1-g}$, and when she plays D instead, it is $U\,(D|C) = 3\frac{1-2g}{1-g} - \frac{g}{1-g}$. In order for it to be an equilibrium, it has to be $U\,(C|C) \geq U\,(D|C)$, which yields $g \geq \frac{1}{3}$. Setting $g = \frac{1}{2}$, (maximum possible, since $f = g$), we obtain the correlated equilibrium with payoffs $1\frac{2}{3}$. We could do similar analysis in general (both players have to be considered), to obtain that e, f, g, and h have to satisfy the inequalities $g \geq e$, $f \geq h$, $f \geq e$, and $g \geq h$, with the constraint that $e + f + g + h = 1$. The players' payoffs will be smallest when $e = 0$, by symmetry set $f = g$. Then the maximal h can be is $\frac{1}{3}$. This gives the correlated equilibrium with payoffs $\frac{2}{3}$.

Exercise 6.9.31

See solution to 6.9.32.

Exercise 6.9.32

Denote the probabilities of different outcomes, and the strategies of the players as in exercise 6.9.30. Doing the same arithmetic with conditional probabilities as in that exercise, we would now obtain the inequalities $g \geq 4e$, $4f \geq h$, $2e \geq f$, and $h \geq 2g$, along with the constraint that $e+f+g+h=1$. We now compute the convex set of possible probabilities. First, it has to be that e, f, g, h are all positive. The reason is that if one were equal to 0, then all would have to be 0, which is a contradiction with the sum being equal to 1. (take for example $e=0$, then by the third inequality $f=0$, so by the second one $h=0$, and by the fourth one $g=0$). Moreover, multiplying the third inequality by 4 and the first one by 2, we can rewrite the four inequalities in a chain: $8e \geq 4f \geq h \geq 2g \geq 8e$. But this implies that all have to be equalities, hence the only correlated equilibrium is $e=\frac{1}{15}$, $f=\frac{2}{15}, g=\frac{4}{15}, h=\frac{8}{15}$, giving the payoffs $\frac{7}{3}$ and $\frac{17}{15}$. Note that this is also the same as the unique mixed Nash equilibrium where I mixes with $\left(\frac{1}{5}, \frac{4}{5}\right)$ and II with $\left(\frac{1}{3}, \frac{2}{3}\right)$.

Exercise 6.9.33

The set of correlated equilibrium outcomes is convex (see answer to 6.9.30) so if it contains each Nash equilibrium outcome it must contain the convex hull of all Nash equilibrium outcomes. If a referee tells Adam and Eve to play (s, t) with $p(s, t)$ then by definition of Nash equilibrium it will be a best reply for both Adam and Eve to follow the instructions (given that the other does follow the instructions). Hence, this is a correlated equilibrium. The same is true if the referee first randomly chooses a Nash equilibrium because referee's choice is independent from the probabilities with which each pair of pure strategies is played in a specific equilibrium.

Exercise 6.9.34

We show that this game has a unique correlated equilibrium, and the uniqueness of Nash equilibrium then follows from the fact that each Nash equilibrium is also a correlated equilibrium (thus if there existed another Nash equilibrium there would also have to exist another correlated equilibrium). Write e, f, g, h for probabilities of recommending pure strategy profiles $(up, left), (up, right), (down, left), (down, right)$, respectively. Now we write down the incentive constraints for player I – that is, I always prefers to play whatever the referee recommends

$$5\frac{e}{e+f} + 1\frac{f}{e+f} \geq 3\frac{e}{e+f} + 2\frac{f}{e+f},$$

$$3\frac{g}{g+h} + 2\frac{h}{g+h} \geq 5\frac{g}{g+h} + 1\frac{h}{g+h}.$$

From here, we get $2e \geq f$ and $h \geq 2g$. Similarly, writing down the incentive constraints for player II, we obtain $g \geq 4e$ and $h \leq 4f$. We can now combine these 4 inequalities, to obtain $2g \geq 8e \geq 4f \geq h \geq 2g$, so that all of these must be equalities. Finally, we use the fact that these probabilities must add up to 1, that is, $e + f + g + h = 1$, to obtain the unique correlated equilibrium which is also the unique Nash equilibrium.

Exercise 6.9.35

We find the symmetric equilibrium, where both players mix with the same probabilities. Denote by F_i the probability that a player bid i cents or less, and by p_i the probability with which a player bid exactly i cents. A player has to be indifferent between any of her possible bids, in particular, she is indifferent between bidding i cents and bidding 1 cent. Her expected payoff from bidding i is $p_i \left(F_{i-1} \left(100 - i\right) - \left(1 - F_{i-1}\right) i\right)$ and her expected payoff of bidding 1 is $F_1 \left(-1\right)$. So these two expressions have to be equal, i.e.

$$p_i \left(F_{i-1} \left(100 - i\right) - \left(1 - F_{i-1}\right) i\right) = F_1 \left(-1\right)$$

Also, $p_i = F_i - F_{i-1}$ and $F_{100} = 1$, since only bids between 1 and 100 cents are allowed. This gives the recursive system of equations $\left(F_i - F_{i-1}\right) \left(F_{i-1} \left(100 - i\right) - \left(1 - F_{i-1}\right) i\right) = -F_1$ and $F_{100} = 1$. Now we need a little guessing, so we check whether $p_i = \frac{1}{100}$ solves this system of equations, and indeed it does. Thus, $p_i = \frac{1}{100}$ is the mixed strategy equilibrium.

Exercise 6.9.36

Suppose player I was instructed to use a strongly dominated strategy, say strategy 1. But then, regardless of what strategy the other player was instructed to use, it is better for player I to use the strategy that strongly dominates strategy 1. Hence, player I would never fulfill such an instruction.

Exercise 6.9.37

If Adam and Eve both knew what referee knows when they are both instructed to play *slow*, they would no longer do it. It is simple to see that: if Adam knows Eve is instructed to play *slow*, then his best reply is to play $fast$; similarly for Eve. Thus, in this case only either of the two Nash equilibria can be played, and Adam and Eve can no longer achieve the payoff $\left(1\frac{2}{3}, 1\frac{2}{3}\right)$.

Exercise 6.9.38

(a) If readers coordinate, then the expected payoff to each one of them is $\frac{1}{n}\$m$, which is strictly better for each one of them than if they all enter, and each obtains the payoff of $\frac{1}{n^2}\$m$.

(b) The probability that all n readers will enter is p^n, and the expected payoff to a reader is $p\left(\sum_{i=0}^{n-1} p^i (1-p)^{n-i} \frac{i}{i+1}\right)$.

(c) Derive the expected payoff obtained in part (b) with respect to p to obtain that the optimal p solves the equation

$$\sum_{i=1}^{n}\left(p^{i-1}(1-p)^{n-i-1}\right)\frac{i}{1+i}(i-np)=0.$$

For instance, when $n=30$, one can use a computer to find out that the optimal $p=0.035$. The probability that no prize is awarded at all is $(1-p)^n$, in the previous example this is approximately 0.34.

(d) In the case where one of the readers is selected at random, it is optimal for any reader who wasn't selected to enter anyway. In the case where socially optimal probability of entering is selected, it is optimal for any of the readers to deviate from such mixed strategy and enter with probability 1 instead. This is so because by changing his own probability of entering, a reader doesn't affect the entering probabilities of the other readers. Since by not entering a reader receives a payoff of 0 and by entering a positive expected payoff, entering is a strongly dominant strategy.

Exercise 6.9.39

No rational agent would be willing to pay any money.

Exercise 6.9.40

Eve wants to match Adam's choice of H (for Heads) or T (for Tails). If Eve matches correctly she obtains 1 and Adam gets 0, and reverse if she doesn't match Adam's choice. Let p be the probability with which Adam chooses H and let σ_h be the probability of Eve choosing H given a signal h. Let $u(Y \mid x)$ denote Eve's payoff if she chooses action Y conditional on signal x. For instance, if she observes a signal h and plays action H, her payoff equals the conditional probability that Adam played H given she observed h, hence

$$u(H \mid h) = \frac{ph}{ph + (1-t)(1-p)}.$$

Similarly,

$$u(T \mid h) = \frac{(1-t)(1-p)}{ph + (1-t)(1-p)},$$

$$u(T \mid t) = \frac{t(1-p)}{p(1-h) + t(1-p)},$$

$$u(H \mid t) = \frac{p(1-h)}{p(1-h) + t(1-p)}.$$

Adam's payoff if he plays H is $u_A(H) = h(1-\sigma_H) + (1-h)$, and if he plays T, it is $u_A(T) = (1-t)\sigma_H$. He is indifferent between these when $\sigma_H = \frac{1}{1+h-t}$. We can plug this into Eve being indifferent between H and T given h, to get that Adam should mix with probability $p = \frac{1-t}{h+1-t} < \frac{1}{2}$. Now we insert this into $u(T \mid t)$ and $u(H \mid t)$, and since $t + h > 1$ and $h > t$, we have that $\frac{h}{1-h} > \frac{1-t}{t}$, implying that $u(T \mid t) > u(H \mid t)$, so that it is optimal for Eve to play T given t. In this equilibrium Adam wins with probability $ph(1-\sigma_h) + p(1-h) + (1-p)(1-t)\sigma_h$, which can be simplified into $\frac{1-t}{h+1-t} = p < \frac{1}{2}$.

(a) Peeking Pennies is a model of Ellsberg's Paradox in which Adam believes that with probability h, Eve (the experimenter) will correctly predict his guess of ball being red and consequently put fewer (or no) red balls in urn B. Similarly, if Adam is about to guess that the ball is blue.

(b) Eve uses her signal simply as a randomization device, and plays H with probability 1 given h. Thus we are back to matching pennies.

(c) In this case the game becomes a game of complete information in which Adam moves first, and Eve observes his move, before making her prediction. Assuming that Eve can perfectly predict Adam's move before he has made it would tie Adam's choice sets to Eve's *beliefs* about what he was going to do.