Exercise 4.11.1

Denote by A the expanded set of Pandora's alternatives. A maximum over the set A is always at least as big as the maximum over any subset of A. For the second part of the question notice that the set of alternatives, as perceived by the respectable lady, changes once she gets to know that cocaine will be offered at the tea party - the tea party is no longer respectable.

Exercise 4.11.2

This example tells us that one's preferences may also have to include one's care for other people, i.e. altruism or "good manners".

Exercise 4.11.3

Totality says that $a \preceq b$ or $b \preceq a$. "P or Q" means that one and only one of "P and Q", "P and (not Q)", "(not P) and Q" is true. The first of these possibilities yields the definition of $a \sim b$, the second yields $a \prec b$, and the third $a \succ b$.

Exercise 4.11.4

Take $a = b$ in the totality formula $a \preceq b$ or $b \preceq a$.

Exercise 4.11.5

Since totality and transitivity apply to both $\preceq$ and $\succeq$, they must also apply to $\sim$. If $a \prec b$ and $b \prec c$, then $a \preceq c$ by the transitivity of of $\preceq$. But it cannot also be that $c \preceq a$, since we could then use transitivity to show that $b \preceq a$. To show that $\prec$ does not satisfy totality, apply exercise 4.11.3 with $a = b$.

Exercise 4.11.6

By definition, we know that $b \prec c$ or $b \sim c$, and since $a \prec b$ we know that not$(a \sim b)$ is true. If $b \prec c$, then we know from exercise 4.11.5 that $a \prec c$ because $\prec$ is transitive. We also know from exercise 4.11.5 that $\sim$ is transitive, so if $b \sim c$ and not$(a \sim b)$, we have not$(a \sim c)$. Therefore, it must be true that

$$a \prec b \text{ and } b \preceq c \Rightarrow a \prec c.$$

Exercise 4.11.7

The voting outcomes are as follows: between Alice and Bob it is Bob, between Bob and Nobody it is Nobody, and between Nobody and Alice it is Alice. Thus the social preference is $Alice \prec Bob \prec Nobody \prec Alice$, which is not transitive.

Exercise 4.11.8

(a) $g = 1.25$, $v = 5$.

(b) $g = \frac{40}{17}$, $v = \frac{10}{17}$.

Exercise 4.11.9

x	a	b	c	d	e	f
$U(x)$	0	0	$\frac{1}{2}$	$\frac{3}{4}$	1	1
$V(x)$	-100	-100	20	21	1000	1000

Exercise 4.11.10

x	A	B	C	D
$U(x)$	0	1	2	1

Exercise 4.11.11

x	$\mathcal{L}$	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{W}$
$v(x)$	0	0.4	0.8	1

Using the utility function v given in the preceding table, we can compute the expected utilities of the lotteries $\mathbf{L}$ and $\mathbf{M}$ to be 0.55 and 0.61 respectively. Since $\mathcal{E}v(\mathbf{M}) > \mathcal{E}v(\mathbf{L})$, it follows that $\mathbf{L} \prec \mathbf{M}$.

Exercise 4.11.12

With $a < 0$, the person prefers less money to more money. With $a = 0$, the person doesn't care how much money he or she has. When $0 \leq a \leq 1$, $u''(x) \leq 0$, and so the person is risk-averse. When $a \geq 1$, $u''(x) \geq 0$, and so the person is risk-loving. If $a = 2$, $\mathcal{E}u(\mathbf{K}) = 0.01 \times 0^2 + 0.89 \times 1^2 + 0.1 \times 5^2 = 3.39$, where money is counted in units of one million. This has to be compared with the utility $u(1) = 1^2 = 1$ of getting \$1m for certain. Since $3.39 > 1$, the person would prefer participating in the lottery to owning \$1m. The dollar equivalent X of the lottery $\mathbf{K}$ is found by solving the equation $u(x) = \mathcal{E}u(\mathbf{K})$. Thus $X = \sqrt{3.39}$.

Exercise 4.11.13

Each extra dollar corresponds to more utils (units of utility) than the dollar before.

Exercise 4.11.14

(a) Since $u(\$1)$ is a von Neumann and Morgenstern utility function, we know that the utility of a lottery **N** with expected payoff of \$1 (respectively \$9) will have a utility value of 0.1 (respectively 0.9). Since Pandora is risk-averse, getting \$1 (respectively \$9) with certainty will be preferred to the lottery. Mathematically, since Pandora is risk-averse, her utility function will be concave. Hence, $u(\$x) \geq \frac{x}{10} u(\$10) = \frac{x}{10}$.

(b) $\mathcal{E}u(\mathbf{L}) = 0.3u(\$1) + 0.2u(\$9) + 0.1$ and $\mathcal{E}u(\mathbf{M}) = 0.2u(\$1) + 0.1u(\$9) + 0.2$. If Pandora expressed the preference $\mathbf{L} \prec \mathbf{M}$, it would have to be true that $\mathcal{E}u(\mathbf{M}) - \mathcal{E}u(\mathbf{L}) = 0.1 - 0.1[u(\$1) - u(\$9)] > 0$. However, in part (a) we showed that $u(\$1) \geq 0.1$ and $u(\$9) \geq 0.9$. Hence, preferring **M** to **L** violates the von Neumann and Morgenstern rationality axioms.

Exercise 4.11.15

Denote by L_1 the option of having one ticket in both lotteries and by L_2 the option of having two tickets in one lottery. To simplify things assume that the probability of getting the prize with any single ticket in any of the two lotteries is p (that is, there are $\frac{1}{p}$ tickets sold in each lottery, and every ticket is equally likely to hit the prize). Also denote by u_2 Bob's utility from obtaining \$2000, and assume that Bob's utility of obtaining \$0 is 0, and his utility from getting \$1000 is 1. Then L_1 has 4 possible outcomes: Bob wins both prizes, Bob wins one prize but not the other (two outcomes), or Bob wins no prizes. Thus, $L_1 = p^2 u_2 + 2p\,(1-p) + (1-p)^2\,0 = 2pu_1 + p^2\,(u_2 - 2)$. On the other hand, in L_2 Bob wins one prize with probability $2p$. Thus, $L_2 = 2p$. Clearly, as long as $u_2 < 2$, Bob prefers L_2. But this is precisely the condition for Bob being risk averse over money.

Exercise 4.11.16

This implies that $u_2 = u_1 = 1$, thus by previous exercise, Bob prefers L_2.

Exercise 4.11.17

Strategies AAD for I and DDD for II imply that $\frac{1}{2}0 + \frac{1}{2}1 \leq a$ (since I prefers to play D at his last move), and $\frac{1}{5}0 + \frac{4}{5}1 \leq b$ (since I prefers to play D at his first move). Thus, $\frac{1}{2} \leq a$ and $\frac{4}{5} \leq b$. Hence, both soldiers are risk averse in this case.

Exercise 4.11.18

Denote by (a_1, a_2, a_3) the vector of the numbers that result in spinning the 3 wheels (i.e. a_1 is the number on which the first wheel stops). Referring to Figure 3.19, observe that for the numbers on the wheels to sum to 15, there are only the following possibilities: $(9,1,5)$, $(4,6,5)$, $(4,8,3)$, $(2,6,7)$, and $(2,8,5)$. Assume that each of the five possible outcomes is equally likely. Now the tree can easily be drawn. First the nature moves and picks each of the outcomes with equal probability. Then, player I moves without observing nature's move, and finally (he has one information set, and three possible actions). Finally, player II moves after observing I's move, but not the move of the nature. Thus, II has 3 information sets, and in each information set he has 2 actions available.

Exercise 4.11.19

Exercise 3.11.32 showed us that the expected difference between the numbers is 0. Hence, a risk-neutral player will not care which wheel he chooses.

Exercise 4.11.20

The six possible events of this type are $(\mathbf{L_1}, \mathbf{L_2})$, $(\mathbf{L_1}, \mathbf{L_3})$, $(\mathbf{L_2}, \mathbf{L_1})$, $(\mathbf{L_3}, \mathbf{L_1})$, $(\mathbf{L_2}, \mathbf{L_3})$, and $(\mathbf{L_3}, \mathbf{L_2})$. For each event, the expected payoff is 0, and he assigns a utility of 1 to a payoff of 0. The expected utility for each of the events is as follows:

$$u(\mathbf{L_1}, \mathbf{L_2}) = 735.68 \qquad u(\mathbf{L_1}, \mathbf{L_3}) = 91.42$$
$$u(\mathbf{L_2}, \mathbf{L_1}) = 91.08 \qquad u(\mathbf{L_3}, \mathbf{L_1}) = 33.72$$
$$u(\mathbf{L_2}, \mathbf{L_3}) = 33.72 \qquad u(\mathbf{L_3}, \mathbf{L_2}) = 91.42$$

Player I is risk-averse, Player II is not risk-averse.

Exercise 4.11.21

The organizer's utility function u is strictly increasing when $u'(x) > 0$ for all x. It is strictly concave when $u''(x) < 0$ for all x. The function u' is strictly decreasing because its derivative is u''.

(a) $M - p(y-z)$. "Actuarially fair" means that the insurance company gets zero expected profit.

(b) Her utility is $u(y - Mf)$ if it is sunny and $u(z + (y-z)f - Mf)$ if it rains. Differentiating her expected utility yields

$$E = -M(1-p)u'(y - Mf) + p(y - z - M)u'(z + (y-z)f - Mf).$$

(c) and (d) If $f = 1$, then $E = 0$ if and only if $M = p(y-z)$.

(e) If $f \geq 1$, then $s = y - Mf \leq z + (y-z)f - Mf = r$. Hence $u'(s) \geq u'(r)$, and so $E \leq u'(r)\{-M(1-p) + p(y-z-M)\} = u'(r)\{-M + p(y-z)\} < 0$.

Exercise 4.11.22

Yes, they are still inconsistent.

Exercise 4.11.23

Let X be the event {pay \$10m and alive}, let Y be the event {die} (if the rich man dies it is irrelevant to him whether he payed \$10m or not), and let W be the event {don't pay and alive}. The statement of the exercise is $\frac{1}{3}u(Y) + \frac{2}{3}u(W) = u(X)$.

(a) Dividing both sides by 2 and adding $\frac{1}{2}u(Y)$ on both sides we obtain $\frac{2}{3}u(Y) + \frac{1}{3}u(W) = \frac{1}{2}\left(u(X) + u(Y)\right)$, which is precisely the other indifference.

(b) $\frac{1}{6}u(Y) + \frac{5}{6}u(W) > \frac{1}{3}u(Y) + \frac{2}{3}u(W) = u(X)$.

Exercise 4.11.24

(a) No.

(b) She will necessarily regret whatever choice she makes.

(c) Yes.

(d) The billionaire has arranged that an integer j be chosen with probability p_j. When Pandora opens her box and finds \$$2^k$, she knows that j is either $k-1$ or k. The event $B = \{k-1, k\}$ has therefore occurred. The event that the other box contains \$$2^{k+1}$ is $A = \{k\}$. The conditional probability we need is therefore

$$\operatorname{prob}(A|B) = \frac{\operatorname{prob}(A \cap B)}{\operatorname{prob}(B)} = \frac{\operatorname{prob}(A)}{\operatorname{prob}(B)} = \frac{p_k}{(p_{k-1} + p_k)}.$$

(e) If the billionaire were right, then $\frac{p_k}{(p_{k-1}+p_k)} = \frac{1}{2}$ for all k. But then all the probabilities p_j are all equal, and so they cannot sum to 1.

Exercise 4.11.25

If Pandora opens her box and finds $\$M_1$ inside, she will certainly regret her choice because $M_1 < M_2$. If she finds $\$M_k$ with $k \geq 2$, she will assign probability $p_{k-1}/(p_{k-1}+p_k)$ to the event that the other box contains $\$M_{k-1}$ and $p_k/(p_{k-1}+p_k)$ to the event that it contains $\$M_{k+1}$. Her expected value for the contents of the other box is therefore $\$(M_{k-1}p_{k-1} + M_{K+1}p_k)/(p_{k-1}+p_k)$. If she regrets her choice, this must exceed $\$M_k$. For the last sentence of the question, simply substitute the values given for M_k and p_k into the inequality.

Exercise 4.11.26

The formula is justified by computing the probability that Pandora will get M_k. When $k \geq 2$, she can get M_k only when the billionaire selects k or $k-1$. Her probability of getting M_k conditional on his choosing k is $\frac{1}{2}$, and it is also $\frac{1}{2}$ conditional on his choosing $k-1$. The total probability of her getting M_k is therefore $\frac{1}{2}(p_{k-1}+p_k)$.

If her initial expected utility is finite, it makes sense to sum both sides of the formula of exercise 4.11.25 between K and infinity, and then to cancel identical terms on each side.[1] This leaves $M_{K-1}p_{K-1} > M_K p_{K-1}$. However, the billionaire needs $M_2 > M_1$ to work his trick. The exercise shows that the von Neumann and Morgenstern theory does not eliminate all problems of the type exposed by the St. Petersburgh paradox.

[1] The sums need to add up to something finite for this canceling to be legitimate. One cannot, for example, deduce that $1 = 0$ from the fact that $1 + 1 + 1 + \cdots = 0 + 1 + 1 + \ldots$.

Exercise 4.11.27

Let probabilities $p_k > 0$ be given. Having chosen $M_1, M_2, \ldots, M_k$ suitably, the billionaire needs to choose M_{k+1} to satisfy the formula of exercise 4.11.25 if he is to be able to play his trick. If Pandora's Von Neumann and Morgenstern utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is unbounded, his task is easy since he need only find a sum of money \$$X$ that makes $M_{k+1} = u(X)$ sufficiently large. Pandora cannot always be risk-loving because u cannot be strictly increasing and bounded if it is concave.

Exercise 4.11.28

If every day in hell brings a disutility of 0, then also the eternity in hell ($\sum_{day=0}^{\infty} 0$) amounts to 0 disutility. There are two possible ways around this. One is to define the disutility of eternity in hell to be $-\infty$, and the disutility of one day in hell to be -1 (or any negative number). The other possibilty is to say that Pandora values present more than the future, that is anything she gets tomorrow is discounted by a factor $\delta < 1$, compared to what she gets today. Then the disutility of one more day is no longer negligible compared to the eternity in Hell.

Exercise 4.11.29

If there is an outcome with an infinite utility (or disutility) then the Postulate 2 of Von Neumann and Morgenstern is violated.