Exercise 3.11.1

In this case it is clear that a lot of information was revealed.

Exercise 3.11.2

Now the Mad Hatter always has another strategy 0 (denoting the strategy of not opening any boxes). If the Mad Hatter chooses 0 whenever this would make Alice lose by switching then Alice is better off by not switching whenever she observes the choice of 0 by the Mad Hatter.

Exercise 3.11.3

The first card in your hand can be chosen in 52 ways. This leaves 51 choices for your second card, and so on. The number of ways in which you can be dealt a hand is therefore $52 \times 51 \times 50 \times 49 \times 48$. But this calculation takes into account the *order* in which the cards are dealt. To find the answer to the question, it is therefore necessary to divide by the number of ways that the five cards you are dealt can be reshuffled into a different order. There are 5 places to which the first card you are dealt may be relocated. This leaves 4 places to which the second card may be relocated, 3 places for the third, and so on. The number of ways of reshuffling your hand is therefore $5 \times 4 \times 3 \times 2 \times 1$. The total number of possible hands is therefore $N = 52 \times 51 \times 50 \times 49 \times 48/5 \times 4 \times 3 \times 2 \times 1$. There are only 4 royal flushes, one for each suit. Thus the probability of being dealt a royal flush in a fair deal is $4/N$.

Exercise 3.11.4

$prob(\heartsuit J) = \frac{1}{47}$ and $prob(Straight) = \frac{4}{47}$.

Exercise 3.11.5

No. Since he is prepared to bet on Punter's Folly, he believes that the probability that Punter' Folly will win is at least $\frac{1}{3}$. Similarly, he believes that the probability that Gambler's Ruin will win is at least $\frac{1}{4}$. Therefore, he must believe that the probability that both will win is at least $\frac{1}{12}$. Note that a more careful answer would be qualified by some discussion of risk-averse or risk-loving preferences. In brief, one then needs the utility functions to be smooth, and to confine attention to small bets.

Exercise 3.11.6

$\frac{5}{12}$.

Exercise 3.11.7

The game with only chance moves can be represented as the following lottery if the chance moves are independent:

L =

$\mathcal{W}$	$\mathcal{L}$
$\frac{1}{2}$	$\frac{1}{2}$

If the chance moves are not independent the situation can be modeled as the following lottery:

M =

$\mathcal{W}$	$\mathcal{L}$
1	0

Exercise 3.11.8

(a) $prob(x = a) = 0.1$.

(b) $prob(y = c) = 0.01$.

(c) $prob(x = a \text{ and } y = c) = 0.01$.

(d) $prob(x = a \text{ or } y = c) = 0.1$.

(e) $prob(x = a \mid y = c) = 1$.

(f) $prob(y = c \mid x = a) = 0.1$.

Exercise 3.11.9

The expected frequency of girls per family is $\frac{1}{2}$.

Exercise 3.11.10

Count all the possibilities in Figure 3.4. to find out that in both cases $prob\,(\text{A pair of queens}) = \frac{1}{3}$.

Exercise 3.11.11

The $prob$ (A pair of queens|Red queen) $= \frac{3}{5}$ and the $prob$ (A pair of queens|A queen of hearts) $= \frac{2}{3}$.

Exercise 3.11.12

There are four possible events: GG, GB, BG, BB (GB means that the older child is a girl and the younger a boy). Assume that all are equally likely. Then $prob\,(\text{two girls}|\text{one girl}) = \frac{1}{3}$ and $prob\,(\text{two girls}|\text{older is a girl}) = \frac{1}{2}$ (count the possibilities).

Exercise 3.11.13

He will need to have started with 2^{19} sequins, which is slightly more than 500.000 sequins.

Exercise 3.11.14

To prove that the lottery Alice is facing has expected dollar value 0 we proceed by induction on the number of coin tosses. Denote by w_m Alice's wealth after m tosses of the coin. Thus, her initial wealth is $w_0 = 31$. Before the first toss of the coin, Alice is facing a lottery of $(1, -1)$ with equal probabilities. So the expected value of the lottery after 1 toss is 0. If after m tosses Alice's wealth is w_m, and Alice has lost in the last n tosses, then she is facing a lottery $(2^n, -2^n)$ with equal probabilities. Thus, conditional on reaching the event at which Alice faces this lottery, the expectation of the lottery is 0. By induction hypothesis the expected dollar value of any lottery with $m-1$ tosses of the coin is 0. But the expected dollar value of the lottery with m tosses will be the same as if we take a lottery where after $m-1$ tosses we consider the expected dollar value of the lottery of one more toss at every event. Since at every event this expected dollar value is equal to the current value of the lottery with $m-1$ tosses, by induction hypothesis the expected dollar value of every lottery with m tosses is 0.

Exercise 3.11.15

The first part is immediate. After betting on the coin toss, when Alice has $n+1$ dollars, she will end up with $n+m+1$ dollars with probability q, and with n, dollars with probability $(1-q)$, which is precisely the equality $p_{n+1} = qp_{n+m+1} + (1-q)\,p_n$. For the second part, observe that $p_n = \frac{1-r^n}{1-r^{s+w}}$ solves the difference equation, with the boundary conditions $p_{s+w} = 1$ and $p_0 = 0$ (to see this just write out the difference equation and manipulate it to obtain that for this form of p_n it is satisfied). Thus $p_s = \frac{1-r^s}{1-r^{s+w}}$.

Exercise 3.11.16

(a) See figure 3.11.16.

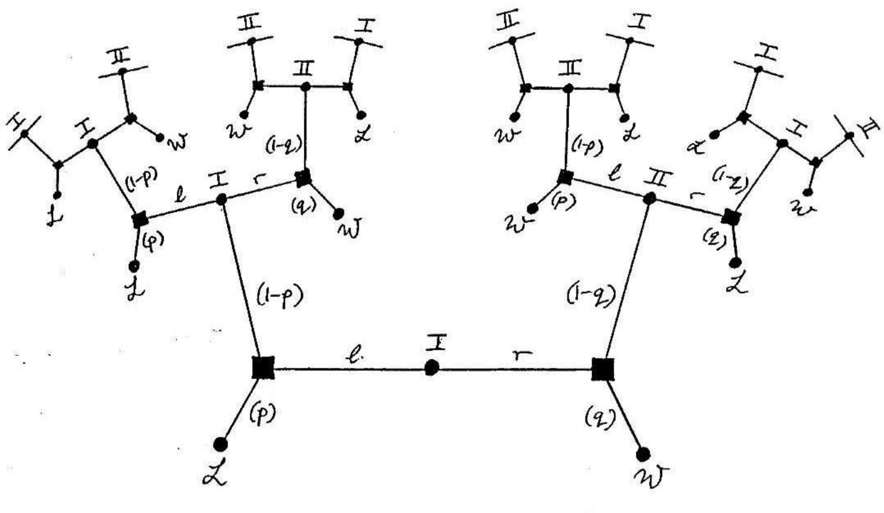

Figure 3.11.16

(b) This is an infinite game because it has an infinite number of nodes.

(c) If $p > 0$, the probability that the game continues forever is 0.

(d) Player I will assure himself of losing by always choosing l. Hence, the only chance he has of winning is to choose r. If he does, he will win with the probability q immediately. If he doesn't win immediately, his probability of winning is $(1 - q)$ times the probability that player II loses. Hence, $v = q + (1 - q)(1 - v)$, and solving for v we obtain $v = 1/(2 - q)$.

(e) If $q = \frac{1}{2}$, then $v = \frac{2}{3}$.

Exercise 3.11.17

Since each player always has an equal chance of being put into a winning position, $v = \frac{1}{2}$.

Exercise 3.11.18

By the same argument as in Section 3.7.1., it has to be that $p(t) = 1 - p(t)$. Solving this equation for t gives $t = 69.3$ days.

Exercise 3.11.19

Suppose that $1 - p_1 > p_2$. Denote by $\tilde{p}_1$ her probability of winning if Alice waits a small amount of time, going on the market at $\tilde{t}$. By continuity of both $p_1(t)$ and $p_2(t)$ there exists a small amount of time such that $\tilde{p}_1 > p_1$ and $1 - \tilde{p}_1 > \tilde{p}_2$. This implies that the probability of Alice winning is higher at $\tilde{t}$, thus her going on the market at t can't be optimal.

Exercise 3.11.20

$d^* = D/\sqrt{2}$.

Exercise 3.11.21

If $p_1(c) + p_2(d) < 1$, then $p_1(c) < 1 - p_2(d)$. Therefore, player II will shoot at d. If $p_1(d) + p_2(c) = 1$, then player I is indifferent between shooting and waiting at d.

Exercise 3.11.22

If $p_1(D) + p_2(D) > 1$, then both players will shoot at the first oportunity. If $p_1(0) + p_2(0) < 1$, then the players will wait to shoot until they are right next to each other. If $p_1(d) + p_2(d) = 1$ for all d satisfying $\frac{1}{3}D \leq d \leq \frac{2}{3}D$, then the players will be indifferent about shooting as soon as they get within $\frac{2}{3}D$ of each other, and they will remain indifferent until they get closer than $\frac{1}{3}D$ to each other.

Exercise 3.11.23

If the player would shoot at d_k in the original game, he would now wait until $d_k + \Delta$, where Δ is the distance between d_k and the node nearest d_{k+1}.

Exercise 3.11.24

Player I will fire at d when $c < d$ if $p_1(d) < .5p_1(c) + .5(1 - p2(c))$. So, player I will shoot if $p_1(d) - .5p_1(c) + .5p_2(c) \geq .5$. If $p_1(x) = p_2(x)$ for an $x \in [0, D]$, then the players will shoot as soon as they have a probability of killing the opponent of at least .5.

Exercise 3.11.25

This is very similar to the 4 steps in the text. The times, at which the increments happen, do not matter. All that matters is who is the next player to get the probability incremented. Assume that $p_1+p_2 \leq \frac{2}{3}$ and Alice is the one to move first. If she waits, her probability of winning is $\frac{1}{2}\left(p_1+\frac{1}{3}\right)+\frac{1}{2}\left(1-p_2-\frac{1}{3}\right)=\frac{1}{2}\left(p_1+1-p_2\right)>p_1$. If $p_1+p_2=1$ and Alice is the one to move first, then if Alice goes on the market, her probability of winning will be p_1. If she waits, her probability of winning will be $\frac{1}{2}\left(p_1+\frac{1}{3}\right)+\frac{1}{2}\left(1-p_2-\frac{1}{3}\right)=\frac{1}{2}\left(p_1+1-p_2\right)=p_1$. On the other hand, if $p_1+p_2 \geq \frac{4}{3}$ then Alice could gain by moving earlier.

Exercise 3.11.26

$prob = \frac{5}{32}$.

Exercise 3.11.27

Step 5. Suppose $f < \frac{1}{2}$. Then player I can improve his probability of winning by waiting in all circumstances. Specifically, if he doesn't move after his toss of the coin, his expected payoff would be $1 - f$. However, player II can use the same logic and not move, so the winner would be determined by the toss of a coin (hence $f \geq \frac{1}{2}$).

Step 3. Suppose it is optimal for player I to move when he gets a "tail". His probability of winning from step 2 is $\frac{1}{6}$. He can steal II's strategy by not moving and allowing her to take her turn. In this case player I will be in a position to lose with certainty if player II gets a "head" and win with probability $\frac{5}{6}$ if player II gets a "tail." Hence, player I's probability of winning is $\frac{5}{12}$ by choosing not to move with a "tail."

Exercise 3.11.28

This possibility is not relevant because it is never optimal for both players to choose not to move before the coin is tossed. Certainly, the game may stop with both players not moving, but we don't need to consider that explicitly because it is considered when we calculate the values of a through e.

Exercise 3.11.29

$v = \frac{2}{3}$.

Exercise 3.11.30

$v = \frac{49}{78}$.

Exercise 3.11.31

(a) $v(p) = \frac{5}{9}$.

(b) See figures 3.11.31(a) and 3.11.31(b) . Each of the lotteries at a terminal node simplifies a structure of the type shown in figure 3.11.31(a).

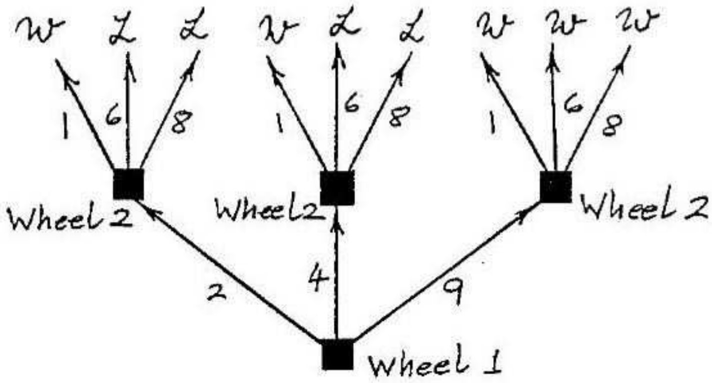

Figure 3.11.31(a)

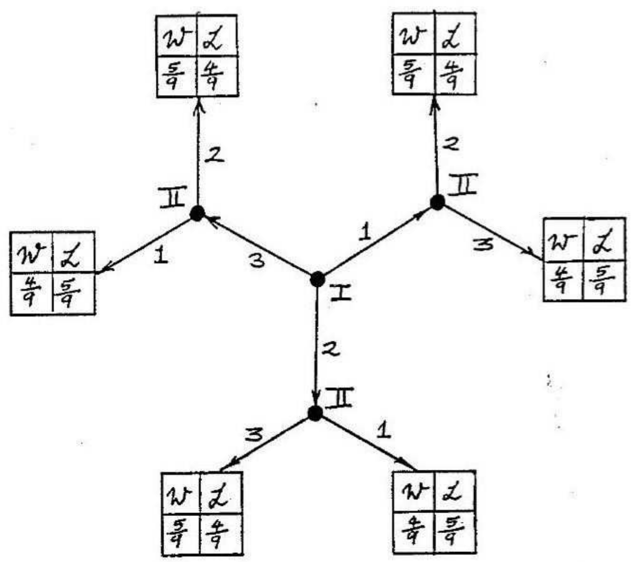

Figure 3.11.31(b)

(c) See figure 3.11.31(b).

(d) We can see this from the game tree in figure 3.11.31(b). If player I chooses wheel 1, player II will choose wheel 3; if player I chooses wheel 2, player II will choose wheel 1; and if player I chooses wheel 3, player II will choose wheel 1.

(e) There is no "best" wheel for player I. What is best for player I depends on what player II decides to do, and she decides after seeing his choice. Player II can choose the wheel that maximizes her expected payoff conditional on what player I does. Hence, going first offers no advantage.

Exercise 3.11.32

$\mathbf{L_2} =$

8	1	6
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$\mathcal{E}\mathbf{L_1} = \frac{1}{3}(9+2+4) = \mathcal{E}\mathbf{L_2} = \frac{1}{3}(8+1+6) = \mathcal{E}\mathbf{L_3} = \frac{1}{3}(7+3+5) = 5.$$

Also, $\text{prob}(\omega_1 - \omega_2 = -2) = \frac{2}{9}$. $\mathcal{E}\mathbf{L_1} - \mathbf{L_2} = \mathcal{E}\mathbf{L_1} - \mathcal{E}\mathbf{L_2}$ because the expectation operator is linear. $\mathcal{E}(\mathbf{L_1} - \mathbf{L_2}) = \mathcal{E}(\mathbf{L_2} - \mathbf{L_3}) = \mathcal{E}(\mathbf{L_1} - \mathbf{L_3}) = \mathcal{E}\mathbf{L_1} - \mathcal{E}\mathbf{L_2} = \mathcal{E}\mathbf{L_2} - \mathcal{E}\mathbf{L_3} = \mathcal{E}\mathbf{L_1} - \mathcal{E}\mathbf{L_3} = 5 - 5 = 0$.

Exercise 3.11.33

(a) If player I chooses the first wheel and player II chooses the second wheel, there are 16 equally likely outcomes. Player I wins in 8 of these, player II wins in 7 of these, and they spin again in one. Hence, $p = \frac{8}{16} + \frac{1}{16}p$.

(b) If both players choose optimally, player I wins with probability $\frac{8}{15}$.

Exercise 3.11.34

(a) There are 6 diamonds in the opponents' hands, and since diamonds are evenly split, the number of all possible opponents' hands is $\frac{6\times5\times4}{3\times2\times1} = 20$. There are 4 hands in which South holds $\diamondsuit K, Q$, so the probability of this event is $\frac{1}{5}$. This is precisely the event in which the first diamond finesse succeeds. This probability is not $\frac{1}{4}$ because conditionally on evenly split diamonds, the events of S holding $\diamondsuit Q$ and $\diamondsuit K$ are not independent. If there were 100 cards in a suit then $prob(Q \mid K, \text{evenly split})$ is nearly $\frac{1}{2}$.

(b) West's probability of winning at least one finesse in diamonds is equal to the probability that North doesn't hold both $\diamondsuit K, Q$, which is by (a) $1 - \frac{1}{5} = \frac{4}{5}$. Note that the probability of South or North holding the $\heartsuit K$ is $\frac{1}{2}$, independent of how diamonds are distributed. The probability that at least one diamond or heart finesse succeeds is thus $1 - prob(\text{neither diamond nor heart succeeds})$. Thus, $prob(\text{neither diamond nor heart succeeds}) = prob(\text{South holds } \heartsuit K) \times (1 - prob(\text{North holds} \diamondsuit K, Q)) = \frac{1}{2} \times \left(1 - \frac{1}{5}\right) = \frac{2}{5}$.

(c) This is the conditional probability that North holds exactly one of $\diamondsuit K, Q$, conditional on holding *at least one*, which is $\frac{3}{5}/\frac{4}{5} = \frac{3}{4}$. The probability of North holding exactly one is the probability that he holds at least one (which is equal to 1− prob(South holds both)) minus the probability that he holds both. The second part follows from the fact that hearts are independent of diamonds.

(d) If there were 100 cards per suit, then $prob(\text{North holds exactly one of} \diamondsuit K, Q) \approx \frac{1}{2}$ and $prob(\text{North holds at least one of} \diamondsuit K, Q) \approx \frac{1}{2}$.

(e) Let $p = prob(\text{play } \diamondsuit K \mid \text{hold } \diamondsuit K, Q)$. We can use Bayes law to write $prob(\text{hold } \diamondsuit K, Q \mid \text{play } \diamondsuit K) = \frac{prob(\text{hold } \diamondsuit K,Q \text{ and play } \diamondsuit K)}{prob(\text{play } \diamondsuit K)}$.

Now, $prob(\text{play } \diamondsuit K) = prob(\text{hold } \diamondsuit K, Q \text{ and play } \diamondsuit K) + prob(\text{hold } \diamondsuit K \text{ and play } \diamondsuit K) = prob(\text{hold } \diamondsuit K, Q \mid \text{play } \diamondsuit K) \times prob(\text{hold } \diamondsuit K, Q) + prob(\text{hold } \diamondsuit K)$. As in part (a), we can obtain that $prob(\text{hold } \diamondsuit K) = \frac{3}{10}$. We can then express $prob(\text{hold } \diamondsuit K, Q \mid \text{play } \diamondsuit K)$ to obtain that it equals to $\frac{p \times 1/5}{p \times 1/5 + 3/10} = \frac{2p}{2p+3}$. The probability that the second diamond finesse succeeds is then $1 - \frac{2p}{2p+3}$.

Exercise 3.11.35

If the players are uninformed, then playing $(Dove, Dove)$ is the only Nash equilibrium of the game, giving each player a payoff 1. If the players are informed of the roll of the die, then whenever 6 is rolled, they would play $(Hawk, Hawk)$; the rest of the times they would still play $(Dove, Dove)$. Their resulting expected payoffs would be $\frac{5}{6}$.

Exercise 3.11.36

No, the story simply implies that Mannie had a very large initial wealth (note the qualifier "rarely").

Exercise 3.11.37

Either the money with which the father started gambling was already enough to pay for the college or there must have been irrational players at the table.