Exercise 14.9.1

A counterfactual would be the line of thought where Adam supposed that the test wasn't given on Monday, and then tried see what the implications were. In the argument of Section 2.3.1 Adam never considers a possibility of something happening that is not supposed to happen. Rather, he thinks what won't happen. His reasoning therefore isn't counterfactual.

Exercise 14.9.2

Deleting weakly dominated strategies proceeds like this. Step 1. Alice deletes ff, which is weakly dominated by fa. Step 2. Bob deletes oo, oi, io which are all weakly dominated by ii. Step 3. Alice eliminates fa, which is weakly dominated by af. The remaining strategy profiles are (aa, ii) and (af, ii). Clearly, both of these result in play moving along the backward induction path. However, the latter one is not a subgame-perfect equilibrium.

Exercise 14.9.3

Payoffs that work are for instance (there are many other possibilities) $(1, 1)$ if the male bird returns, $(0, 0$ if the male leaves and female searches for a new partner, and $(2, -1)$ if the female continues to incubate the eggs. In the biological context it makes to view backward induction as a trembling-hand equilibrium. The reason is that not all birds are exactly the same, so that even if they have the equilibrium strategies hard-wired in their brains, occasionally a bird may play out-of equilibrium action. In other words, some birds may tremble.

Exercise 14.9.4

Strategy *right* is dominated by $left$ for player 1, and *down* is dominated by *up* for player 3. The two Nash equilibria are $(top, left, up)$ and $(bottom, left, up)$. Clearly, no player uses a dominated strategy in either of the two Nash equilibria. Only the $(top, left, up)$ is trembling-hand equilibrium. The $(bottom, left, up)$ isn't a trembling-hand equilibrium because if either Player 2 trembles by playing *right* with probability ϵ or Player 3 trembles by playing *down* with probability ϵ, it is always optimal for Player 1 to switch to playing *top*, regardless of how tiny the ϵ is.

Exercise 14.9.5

See figure 14.9.5 for the extensive form of the simultaneous-move game. In this game, $(\ell L, r)$ is a trembling-hand equilibrium, and it is not a trembling-hand equilibrium in the original game (it is easy to check that I would want to deviate to ℓR).

Exercise 14.9.6

From Section 13.2.3 we have that $prob(l|R) = \frac{p}{1+p}$ and $prob(r|R) = \frac{1}{1+p}$, where p is Alice's prior that Hatter will open $Box2$. Therefore, Alice's expected gain from playing s is $\frac{1}{1+p}(pL+W)$, and her expected gain from playing S is $\frac{1}{1+p}(pW+L)$, where $W > L$. Thus, it is for her optimal to play s for any p, where $0 \leq p \leq 1$. Finding Hatter's optimal play is now easy. Since Alice will always switch, independently of the probability with which Mad Hatter mixes between $Box2$ and $Box3$, it also doesn't matter for Hatter how he mixes. In other words, the assessment equilibria are: Hatter opens $Box2$ with probability p, Alice plays s, and her belief is $prob(l|R) = \frac{p}{1+p}$, $prob(r|R) = \frac{1}{1+p}$, for any $p < 1$. When $p = 1$ there is also a continuum of equilibria, where Alice plays s with probability γ, for any $\gamma \in [0, 1]$, and her belief is as before.

Exercise 14.9.7

The strategy profile (fa, oi) is no longer a part of an assessment equilibrium. To see that notice that for ϵ small enough, it is for Bob still optimal to play *in* at the second stage, since $3(1-\epsilon) > 1$. However, it is no longer optimal for Bob to play *out* at the first stage, since $3(1-\epsilon) > 2$ for ϵ small enough. Thus, (oi, af) is no longer an assessment equilibrium. The other assessment equilibrium found in 14.3.1 changes slightly.

Exercise 14.9.8

To check that (d, A, l) is a Nash equilibrium observe that Player 1 doesn't want to deviate because he would then obtain 1 instead of 3, Player 2 is indifferent between deviating or not, and Player 3 would obtain 0 instead of 2, if he deviated. Similarly, in (a, A, r), Player 1 would if he deviated obtain 0 instead of 1, Player 2 would obtain 0 instead of 1, and Player 3 is indifferent between deviating or not. The reason both Nash equilibria are also subgame-perfect equilibria is that the game is in fact a simultaneous-move game. Thus, any Nash eq. will also be a subgame-perfect equilibrium. Now denote Player 3's information set by IS_3, the left node in IS_3 by L, and the right node by R. Assume that Player 3 had a belief $prob(L|IS_3) = p$ and $prob(R|IS_3) = 1 - p$. Then, it is optimal for Player 3 to play l if $p > \frac{1}{3}$, r if $p < \frac{1}{3}$, and he is indifferent if $p = \frac{1}{3}$. Now, if Player 2 at his decision node knew that Player 3 would play l for sure, his optimal action would be D, and if Player 1 knew this, his optimal action would be to play a for sure. But then the $prob(L|IS_3) = 0$, so that $p \geq \frac{1}{3}$ is not a belief that could form an assessment equilibrium with (d, A, l), so that (d, A, l) is not a part of any assessment equilibrium. If $p \leq \frac{1}{3}$ and Player 3 plays r for sure, then it is optimal for Player 1 to play a for sure, and for Player 2 to play A for sure, implying that IS_3 is never reached, so that any belief of Player 3 is possible. In particular, any belief with $p \leq \frac{1}{3}$ is possible, so that any such belief constitutes an assessment equilibrium with (a, A, r).

Exercise 14.9.9

We draw an extensive form for the case where Adam and Eve are independently picked by Nature to be rational with probability $1 - \epsilon$, and each irrational type has the same frequency in the population of Adams as in the population of Eves. See figure 14.9.9.

Must attach figure 14.9.9

Exercise 14.9.10

Let p be the probability with which the robot plays across. We look for an assessment equilibrium in which there is a node k at which Eve moves (where k is an integer, denoting how many moves Adam had prior to that node), such that both Adam and Eve play *across* for sure at all nodes before k, and both play *down* after k. Thus, Alice's payoffs at k are either $x-1+2c$, $x-1+c$, or $x+c$, where $x = 1+k-(k+1)c$. Let q denote the conditional probability that Eve is facing a robot, given that she finds herself at k. Now we have $q = \Pr(robot|k) = \frac{\epsilon p^k}{\epsilon p^k+(1-\epsilon)}$. Alice will then play *across* if

$$x - 1 + 2c \leq q\left[(1-p)(x-1+c) + p(x+c)\right] + (1-q)\left[x-1+c\right],$$

which reduces to $c \leq qp$. Thus, k is the largest integer such that $c \leq p\frac{\epsilon p^k}{\epsilon p^k+(1-\epsilon)}$. It is then indeed optimal for Alice to play *down* at her next node (since then $c > p\frac{\epsilon p^{k+1}}{\epsilon p^{k+1}+(1-\epsilon)}$), hence this is an assessment equilibrium.

Exercise 14.9.11

The assumption that was made is that Adam and Eve are rational, despite having behaved irrationally in the past, since the only way those nodes could possibly be reached is if at least one of Adam and Eve was irrational.

Exercise 14.9.12

The pure strategy Nash equilibria are $(Dt, ll), (Bt, lr), (Dt, rl), (Db, rr)$.

Exercise 14.9.13

The unique Nash equilibrium of the game is (bq, bd). An easy way to find it is to firs eliminate strongly dominated strategies for both players. That is for Player 1, qb is strongly dominated by bq, and for Player 2, dd is strongly dominated by $\frac{1}{2}bd + \frac{1}{2}db$. All the other strategies are rationalizable. In the remaining 3x3 matrix it is pretty easy to see that (bq, bd) is the only Nash equilibrium.

Exercise 14.9.14

The Nash equilibria are (l, L) and (a, R), so that there is an equilibrium selection problem. The equilibrium selected by forward induction is (l, L). The argument goes like this. If Player II were to move in his information set, then he should believe that the only reason the game got there was because Player I played l in order to obtain a payoff of 4, since he could otherwise have played a and obtain 3, rather than 1. In other words, Player I can't be thinking that R will be played, so that Player II should play L. But then Player I will always obtain a payoff of 4 whenever he plays l, which is better than getting 3 from playing a. Thus, Player I should play l, and Player II's best reply is L.

Exercise 14.9.15

	L	R
U	2 1	1 4
D	3 2	4 3

Exercise 14.9.16

Enumerate the strategies of Player 1 by $a1, a2, a3, a4$, and the strategies of Player 2 by $b1, b2, b3, b4$. Elimination proceeds as follows. Step 1. $b4$ strongly dominates $b1$. Step 2. $a2$ strongly dominates $a1$, and $a3$ strongly dominates $a4$. Step 3. $\frac{1}{2}b2 + \frac{1}{2}b3$ strongly dominates $b4$. Step 5. $a2$ strongly dominates $a3$. Step 6. $b2$ strongly dominates $b3$. Thus, the unique rationalizable pair of pure strategies is $(a2, b2)$.

Exercise 14.9.17

(a) We check for Alice's profit, $\pi(a,b) = (K - c - a - b)a$. The easiest way to do that is to check that the second derivative of π with respect to a is negative. That is, $\frac{\partial^2 \pi(a,b)}{\partial a^2} = -2$. (Another way to see that immediately is that Alice's profit is a quadratic function of a with a negative lead coefficient.)

(b) The diagram is Figure 10.1. with only the reaction curves.

(c) First, an intuitive geometrical argument. From the diagram, $(0, R(0))$ is a point that lies on the lower (Bob's) reaction curve. To compute Alice's reaction to $x_1 = R(0)$, we have to draw a horizontal line to the upper (Alice's) reaction curve, which will give us x_2. Proceeding in a similar fashion, we observe that (x_{2n}, x_{2n+1}) always lies on the lower curve, while (x_{2n-1}, x_{2n}) always lies on the upper curve, and to get from (x_{2n-1}, x_{2n}) to (x_{2n}, x_{2n+1}) we have to draw a horizontal line from the lower curve to the upper curve, and to get from (x_{2n}, x_{2n+1}) to (x_{2n+1}, x_{2n+2}) we have to draw a vertical line from the upper curve to the lower curve. From the picture it is then obvious that the points will converge to the point $(\tilde{x}, \tilde{x})$, implying that the sequence $\{x_n\}$ converges to $\tilde{x}$. Clearly, at the point $(\tilde{x}, \tilde{x})$ Bob and Alice are both playing best replies to each other, so that it is a Nash equilibrium. To see that it is unique, notice that it is the unique point at which their reaction curves cross. Analytical argument that $\tilde{x} = R(\tilde{x})$ is roughly as follows. First, $R(.)$ is a continuous function because it is linear. So if $\tilde{x} \neq R(\tilde{x})$ then by the definition of the limit, for any $n \geq N$, x_n will be very close to $\tilde{x}$. But then $R(x_n)$ will be very close to $R(\tilde{x})$, which is some positive distance away from $\tilde{x}$, so that $R(x_n) = x_{n+1}$ is a positive distance away from $\tilde{x}$, whixh is a contradiction of the fact that $\tilde{x}$ was the limit of the sequence.

(d) The most either of them will ever produce is x_1 because this is their monopoly production level, that is, it is the best reply to the other producing 0.

(e) Given that Bob knows that Alice will never produce more than x_1, he will never produce less than $R(x_1) = x_2$, because his reaction curve is decreasing in Alice's output level.

(f) The part of the diagram above x_3: if Bob knows Alice is never going to produce less than x_2 then he should never produce more than his best reply to x_2 which is x_3.

(g) At the $2n-1$-th step Bob will always erase everything below x_{2n} and at the $2n$-th step he will erase everything above x_{2n+1}, so if a point q is not to be erased in the process, it has to satisfy the criterion $x_{2n} \leq q \leq x_{2n+1}$ for all n.

(h) From part (c), the only q that satisfies $x_{2n} \leq q \leq x_{2n+1}$ for all n is the Nash equilibrium. Now, at every step Bob and Alice were in fact deleting strongly dominated strategies, so the Nash equilibrium is the only rationalizable strategy profile.

Exercise 14.9.18

Being a "best reply to some mixed strategy" means that there is some mixed strategy q of Player II, for which playing $\tilde{p}$ will give Player I at least as much as playing any other of all his possible strategies, which is precisely what the first inequality states. To check the equivalence with the second statement, first assume that the first statement holds. This is equivalent to the fact that there exists a strategy q, for which $p^T Aq - \tilde{p}^T Aq \leq 0$ for all $p \in P$, which is $((p - \tilde{p})^T Aq \leq 0$. But since this is true for all $(p - \tilde{p})$ it is also true for the maximum over all p (which is the same as maximum over all $p - \tilde{p}$). Now since the inequality holds for one particular q it also must hold for the minimum over all possible q, so that this direction is proven. To prove the other direction, let the second statement be true. Since the game is finite,it follows that the set of mixed strategies of each player is compact. Now, the function $p^T Aq$ is a continuous function of q for every p, so it follows that $\max_{p \in P_0} p^T Aq$ is attained at some $p(q)$ for every $q \in Q$. Moreover, $p(q)$ is a continuous function of q by the Theorem of the maximum, so that $p(q)^T Aq$ is a continuous function of q and therefore attains a minimum at some $\tilde{q} \in Q$. In other words, $0 \geq \min q \in Q \max_{p \in P_0} p^T Aq = p(\tilde{q})^T A\tilde{q}$. Also, since $p(\tilde{q})$ is by definition the maximizer of $p^T A\tilde{q}$, it follows that $p^T A\tilde{q} \leq p(\tilde{q})^T A\tilde{q} \leq 0$ for every $p \in P_0$. Finally, $0 = p(\tilde{q})$ since $0^T Aq = 0$, and $0 \in P_0$, that is $0 = \tilde{p} - \tilde{p}$, which proves that the maximum in fact has to be attained at $\tilde{p}$, which completes the proof.

Exercise 14.9.19

The first statement just says that for every mixed strategy of Player II, the strategy p gives Player I a strictly better payoff than $\tilde{q}$ which is precisely the definition of strong domination. The second statement is logical negation of the first statement, so that if the second statement holds, the first doesn't, so $\tilde{p}$ is then not strongly dominated. Next, assume first that $\max_{p \in P_0} \min_{q \in Q} p^T Aq \leq 0$. Then, it is also $\min_{q \in Q} p^T Aq \leq 0$ for every $p \in P_0$, and since for every p $p^T Aq$ is a continuous function of q it follows that for every p the $\min_{q \in Q} p^T Aq$ is attained at some $q(p)$. But then $p^T Aq(p) \leq 0$ for every p so that indeed for every p there exists a q such that $p^T Aq \leq 0$, which proves that the third statement implies the second. To prove the other direction, assume that the third statement doesn't hold, so that $\max_{p \in P_0} \min_{q \in Q} p^T Aq > 0$. Since the set of mixed strategies is compact and $p^T Aq$ is continuous in both p and q, it follows that $\max_{p \in P_0} \min q \in Q p^T Aq$ is attained at some $\tilde{p}, \tilde{q}$. But then $\tilde{p}^T Aq \geq \tilde{p}^T A\tilde{q} > 0$ so that statement 1 holds, which implies that statement 2 doesn't hold. Thus, we have proven that "not statement 3" implies "not statement 2" which is the same as saying that statement 2 implies statement 3 which completes the proof.

Exercise 14.9.20

From Von Neumann's minimax theorem we know that $\max_{p \in P} \min q \in Q p^T A q = \min q \in Q \max_{p \in P} p^T A q$, which doesn't change if we replace P with P_0 (in $\tilde{p}^T A q$ is always a constant with respect to p, so in that part we can exchange the order of maximization and minimization without any problems). The rest now follows from 14.9.18 and 14.9.19.

Exercise 14.9.21

$\Pr(d \geq r) = \Pr(2\sqrt{r^2 - x^2} \geq r) = \Pr\left(\frac{\sqrt{3}}{2} \geq \frac{x}{r}\right)$

(a) $\Pr\left(\frac{\sqrt{3}}{2} \geq \frac{x}{r}\right) = \Pr\left(x \leq \frac{\sqrt{3}}{2}r\right) = \frac{\sqrt{3}}{2}$

(b) Now we have $\Pr(x \leq \alpha) = \left(\frac{\alpha}{r}\right)^2$, implying that $\Pr\left(x \leq \frac{\sqrt{3}}{2}r\right) = \frac{3}{4}$.

In order to compute $\Pr(d \leq \epsilon r)$ it matters how d is distributed, in particular, trembles are 2-dimensional so when computing the payoff, one should keep in mind that every point in the area is equally likely.

Exercise 14.9.22

Let k be a random variable given by the length of the arc on the circumference divided by r, so that in this case, k is distributed uniformly on $[0, 2\Pi]$ (by symmetry). Hence, $x = \tan(k)$. Now we have $\Pr\left(x \leq \frac{\sqrt{3}}{2}r\right) = \Pr\left(\tan(k) \leq \frac{\sqrt{3}}{2}r\right) = \Pr\left(k \leq \arctan\left(\sqrt{3}2r\right)\right) > \Pr\left(k \leq \frac{\sqrt{3}}{2}r\right)$. This example points out that the principle of insufficient reason might be interpreted in a variety of different ways, and depending on how we reason the results will be different.