Exercise 12.12.1

(a) $\{3, 5, 7, 8\}$.

(b) $\{1, 2, 3, 5\}$.

(c) $\{5, 6, 7\}$.

Exercise 12.12.2

Axiom K4 says $\sim \mathcal{K} \sim E \subset \mathcal{K}(\sim \mathcal{K}) \sim E$. Now put $\sim E$ in place of E, and apply K2 to obtain $\sim \mathcal{K}E = \mathcal{K}(\sim \mathcal{K})E$ By taking the negation on each side of this equality, we obtain $\mathcal{K}E =\sim (\sim \mathcal{K}E) =\sim \mathcal{K}(\sim \mathcal{K})E = (\sim \mathcal{K})^2 E$. If Aristotle thought he was living in a small world he couldn't have thought that he knew there were secrets he didn't know about. Hence, Aristotle must have thought he lived in a large world.

Exercise 12.12.3

(a) Since $E \subseteq F$, we know $E \cap F = E$. Therefore, using (K1) we have $\mathcal{K}(E \cap F) = \mathcal{K}E \cap \mathcal{K}F = \mathcal{K}E \Rightarrow \mathcal{K}E \subseteq \mathcal{K}F$. This means that if F occurs every time E occurs, then you cannot know E occurred without also knowing that F occurred.

(b) From (K3) we know $\mathcal{K}E \subseteq \mathcal{K}^2 E$. From (K2) we know $\mathcal{K}\mathcal{K}E = \mathcal{K}^2 E \subseteq \mathcal{K}E$. Therefore, $\mathcal{K}E = \mathcal{K}^2 E$. This means that if you know something, you know you know it.

(c) Using (K4) we know $\mathcal{P}(\sim E) \subseteq \mathcal{K}\mathcal{P}(\sim E)$. However, applying the definition of $\mathcal{P}$, we get $\mathcal{P}(\sim E) = \sim \mathcal{K}E \subseteq \mathcal{K}(\sim \mathcal{K})E$. This is equivalent to $\mathcal{K}E \supseteq (\sim \mathcal{K})^2 E$. This means that if you don't know that you don't know something, then you know it.

Exercise 12.12.4

To show the equivalence, we will use (K0) to derive (P0), (K1) to derive (P1), and so on.

Using (K0) we have that $\emptyset = \sim \Omega = \sim \mathcal{K}\Omega = \sim \mathcal{K} \sim \emptyset = \mathcal{P}\emptyset$, by the definition of $\mathcal{P}$.

Using (K1) and the definition of $\mathcal{P}$, we have that $\mathcal{P}(E \cup F) = \sim \mathcal{K} \sim (E \cup F) = \sim \mathcal{K}(\sim E \cap \sim F) = \sim (\mathcal{K} \sim E \cap \mathcal{K} \sim F) = (\sim \mathcal{K} \sim E) \cup (\sim \mathcal{K} \sim F) = \mathcal{P}E \cap \mathcal{P}F$.

Using (K2) we can deduce that:

$$\mathcal{K} \sim E \subseteq \sim E$$

$$\sim \mathcal{K} \sim E \supseteq E$$

$$\mathcal{P}E \supseteq E$$

Using (K3) we can deduce (P3) as follows:

$$\mathcal{K} \sim E \subseteq \mathcal{K}\mathcal{K} \sim E$$

$$\sim \mathcal{K} \sim E \supseteq \sim \mathcal{K}(\mathcal{K} \sim E)$$

$$\mathcal{P}E \supseteq \mathcal{P}(\sim \mathcal{K} \sim E) = \mathcal{P}^2 E.$$

Finally, the equivalence of (K4) and (P4) is easily shown:

$$\mathcal{P} \sim E \subseteq \mathcal{K}\mathcal{P} \sim E$$

$$\sim \mathcal{K}E \subseteq \mathcal{K} \sim (\mathcal{K}E)$$

$$\mathcal{K}E \supseteq \sim \mathcal{K} \sim (\mathcal{K}E)$$

$$\mathcal{K}E \supseteq \mathcal{P}\mathcal{K}E$$

Exercise 12.12.5

(*i*) $E \subseteq F \Rightarrow \mathcal{P}F \supseteq \mathcal{P}E$. This means that if F happens whenever E happens, then if you think E is possible you must think F is possible.

(*ii*) $\mathcal{P}E = \mathcal{P}^2E$. The set of states you think are possible when E occurs is the same as the set of states you think are possible if you think it is possible E occurs.

(*iii*) $(\sim \mathcal{P})^2E \supseteq \mathcal{P}E$. If you think something is possible, then you don't think it is possible that it is not possible.

Exercise 12.12.6

Before the guard speaks, it is not a truism for Alice that everyone has a dirty face because it is true, but she doesn't know it. That is, she can't tell whether she is in state 1 or state 2 before the guard speaks. In the more rigorous terminology used later in the chapter, the communal possibility set is the entire universe, so the only thing that can be common knowledge is that either everyone does or does not have a dirty face.

Exercise 12.12.7

From (K2) we know $\mathcal{K}T \subseteq T$. If T is a truism then by definition $T \subseteq \mathcal{K}T$. Therefore, $T = \mathcal{K}T$. Going the other direction, if $T = \mathcal{K}T$ then it is true that $T \subseteq \mathcal{K}T$. Hence, T is a truism.

Exercise 12.12.8

(a) By (K3), $\mathcal{K}E \subseteq \mathcal{K}(\mathcal{K}E)$. Thus $\mathcal{K}E$ is a truism.

(b) Show that $\sim \mathcal{K}F$ is a truism by writing $E = \sim F$ in (K4) and recalling the definition of $\mathcal{P}$.

(c) Use (K4).

(d) Put $F = \sim E$, and use the fact that $\sim \mathcal{K}F$ is a truism by part (b).

Exercise 12.12.9

Since S is a truism, we know that $S = \mathcal{K}S$ and $\sim S =\sim \mathcal{K}S =\sim \mathcal{K}\mathcal{K}S$ because $\mathcal{K}S$ is a truism from exercise 12.12.8. Applying (K4), the fact that $S = \mathcal{K}S$, and the definition of $\mathcal{P}$ we have $\sim \mathcal{K}\mathcal{K}S =\sim \mathcal{K}S \subseteq \mathcal{K} \sim \mathcal{K}S = \mathcal{K} \sim S \subseteq\sim S =\sim \mathcal{K}S =\sim \mathcal{K}\mathcal{K}S$. Therefore, we have $\sim S = \mathcal{K} \sim S$ and S is a truism. To see that $S \cap T$ is a truism, we must merely note $S = \mathcal{K}S$ and $T = \mathcal{K}T$. Applying (K2) we have $\mathcal{K}(S \cap T) = \mathcal{K}S \cap \mathcal{K}T = S \cap T$. So, $S \cap T$ is a truism. To see that $S \cup T$ is a truism, we must note that $S \cup T =\sim (\sim S \cap \sim T)$. Since S and T are truisms, $\sim S$ and $\sim T$ are truisms. Therefore, $(\sim S \cap \sim T)$ is a truism from earlier in this exercise. It follows that $\sim (\sim S \cap \sim T)$ is a truism, so $S \cup T$ is a truism.

Exercise 12.12.10

The first inclusion is implied by (K2). The second follows from the fact that the set of all E is larger than the set of those E of the form $E = \mathcal{K}F$. The next identity follows from exercise 12.12.3(b). For the final part, one uses the earlier parts to show that

$$\mathcal{P}\{\omega\} = \bigcap_{\omega \in T} T = \bigcap_{\omega \in \mathcal{K}(\mathcal{K}E)} \mathcal{K}E = \bigcap_{\omega \in \mathcal{K}E} \mathcal{K}E$$

by the first part of the exercise.

Exercise 12.12.11

Let $\omega \in \mathcal{K}E$. By theorem 12.3 we have that

$$\mathcal{P}\{\omega\} = \bigcap_{\omega \in T} T = \bigcap_{\omega \in \mathcal{K}F} \mathcal{K}F \subseteq \bigcap_{\omega \in \mathcal{K}F} F \subseteq E.$$

In the above, T is the set of all truisms that contain ω. $\mathcal{K}F$ is the set of events in which you know F occurred, and $E \in \{F : \omega \in \mathcal{K}F\}$. This means $\mathcal{K}E \subseteq \{\omega : \mathcal{P}\{\omega\} \subseteq E\}$. Now let $\omega \in \{\xi : \mathcal{P}\{\xi\} \subseteq E\}$. We know $\omega \in \mathcal{P}\{\omega\} = \mathcal{K}\mathcal{P}\{\omega\}\mathcal{K}E$ because $\mathcal{P}\{\omega\}$ is a truism and $\mathcal{P}\{\omega\} \subseteq E$. Hence, $\{\omega : \mathcal{P}\{\omega\} \subseteq E\} \subseteq \mathcal{K}E$. Putting these together we have $\mathcal{K}E = \{\omega : \mathcal{P}\{\omega\} \subseteq E\}$.

Exercise 12.12.12

See figure 12.12.12

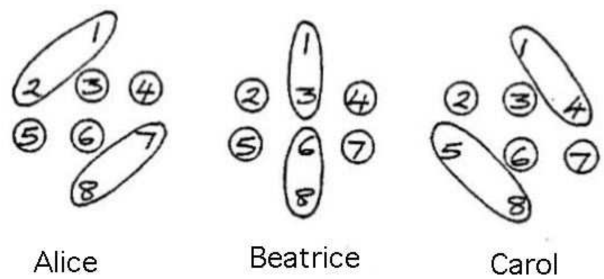


Figure 12.12.12

Exercise 12.12.13

See figure 12.12.13.

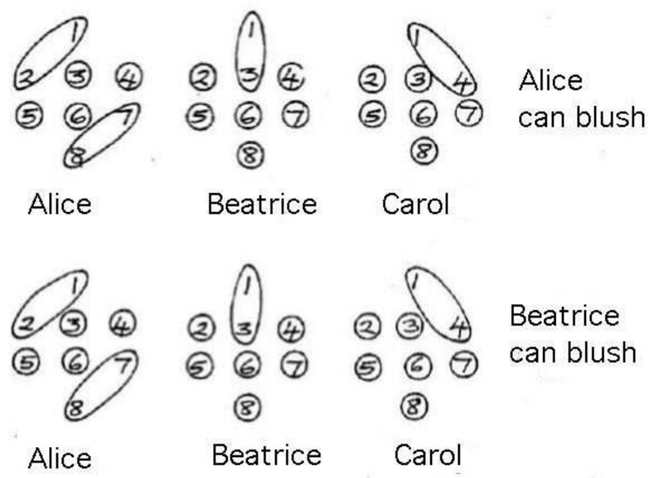


Figure 12.12.13

Exercise 12.12.14

Nobody blushes until the third second, at which time everybody blushes simultaneously. See figure 12.12.14.

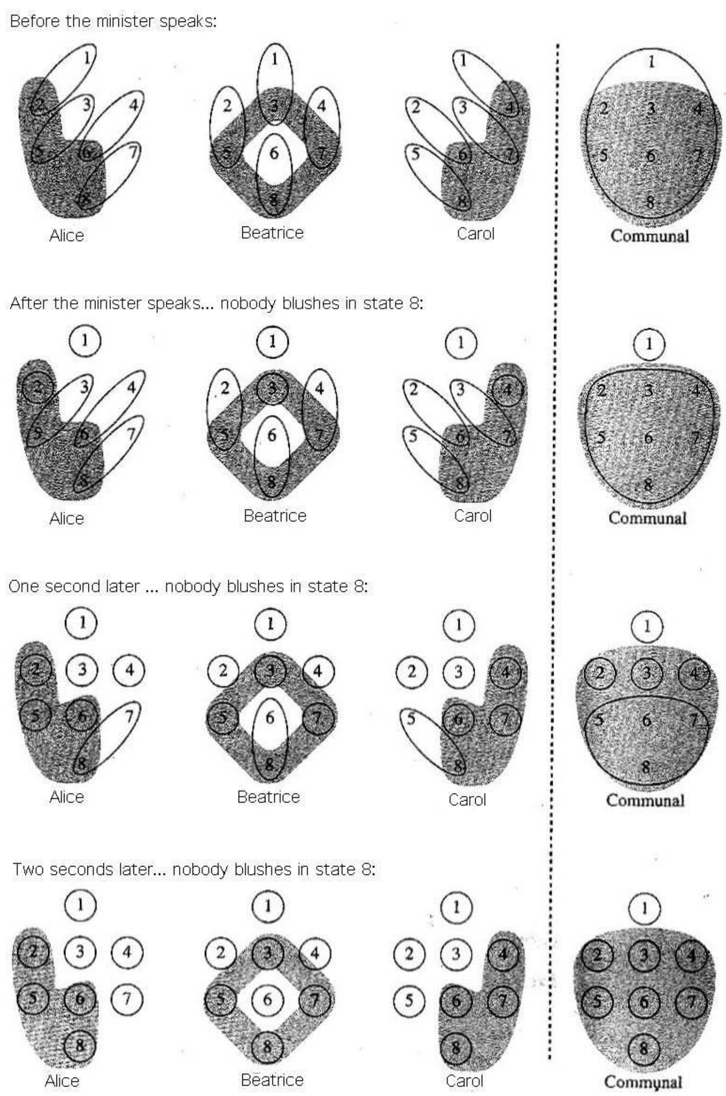


Figure 12.12.14

Exercise 12.12.15

Let Alice have the first opportunity to blush, and then let Beatrice and Carol have the opportunity to blush *simultaneously* one second later. Then let Alice have the opportunity to blush, and so on.

Exercise 12.12.16

(a) The mixed strategy that leads to the same lottery over outcomes as the behavioral strategy in which she assigns equal probabilities at each information set will have her play her pure strategies LLL, LLR, LRR, LRL, RLL, RLR, RRR and RRL with equal probability.

(b) The behavioral strategy is for player II to choose R with probability $\frac{2}{3}$ and L with probability $\frac{1}{3}$ at the first information set. Then she chooses L with certainty at the second information set, and R with certainty at the third information set.

Exercise 12.12.17

(a) It is not a game of perfect information because all of the information sets are not singletons. It is a game of perfect recall because the players know all of their own previous choices.

(b) Player II's behavioral strategy is to choose d at the first information set with probability $\frac{2}{3}$ and u with probability $\frac{1}{3}$. At her second information set, she should choose D with certainty.

Exercise 12.12.18

The mixed strategy is rr with probability pP, rl with $p(1-P)$, lr with $(1-p)P$, and ll with $(1-p)(1-P)$. In order to play rr with $\frac{1}{2}$ and ll with $\frac{1}{2}$, the player would have to play a behavioral strategy which assigns $\frac{1}{2}$ to each of the two strategies in the first information set. In the second information set he would then have to assign probability 1 to each of the two strategies, which is impossible. Kuhn's theorem does not apply because this is a game with imperfect recall.

Exercise 12.12.19

(i) The player gets a payoff of zero for each of his pure strategies.

(ii) Any mixed strategy will yield a convex combination of the payoffs to the pure strategies. Hence, a mixed strategy must have a payoff of zero.

(iii) If the player chooses between l and r with equal probability at the information set containing nodes x and y, he will have an expected payoff of $\frac{1}{4}$. Kuhn's theorem does not apply because the game has imperfect recall.

Exercise 12.12.20

To show $\mathcal{K}$ satisfies (K0) note $\mathcal{K}\Omega = \mathcal{K}_A\Omega \cap \mathcal{K}_B\Omega \cap \mathcal{K}_C\Omega = \Omega \cap \Omega \cap \Omega = \Omega$. To show $\mathcal{K}$ satisfies (K1) notice that $\mathcal{K}(E \cap F) = \mathcal{K}_A(E \cap F) \cap \mathcal{K}_B(E \cap F) \cap \mathcal{K}_C(E \cap F) = \mathcal{K}_A E \cap \mathcal{K}_A F \cap \mathcal{K}_B E \cap \mathcal{K}_B F \cap \mathcal{K}_C E \cap \mathcal{K}_C F = (\mathcal{K}_A E \cap \mathcal{K}_B E \cap \mathcal{K}_C E) \cap (\mathcal{K}_A F \cap \mathcal{K}_B F \cap \mathcal{K}_C F) = \mathcal{K}E \cap \mathcal{K}F$. (K2) is easily shown as follows: $\mathcal{K}E = \mathcal{K}_A E \cap \mathcal{K}_B E \cap \mathcal{K}_C E \subseteq E$. This is true because $\mathcal{K}_i E \subseteq E$ for $i \in \{A, B, N\}$. Alternative examples to that in section 12.6.2 can be found by making a similar argument about states 6 and 7.

Exercise 12.12.21

The operator $\mathcal{K}$ = (somebody knows) should be defined by

$$\mathcal{K}E = \mathcal{K}_A E \cup \mathcal{K}_B E \cup \mathcal{K}_C E.$$

A counterexample will be used to show that (K1) is not satisfied. Suppose only Alice and Beatrice are involved and $E \subseteq F$. Then $\mathcal{K}(E \cap F) = (\mathcal{K}_A(E \cap F)) \cup (\mathcal{K}_B(E \cap F)) = (\mathcal{K}_A E \cap \mathcal{K}_A F) \cup (\mathcal{K}_B E \cap \mathcal{K}_B F) = \mathcal{K}_A E \cup \mathcal{K}_B E \neq \mathcal{K}_A \cap \mathcal{K}_B E$.

Exercise 12.12.22

The operator $\mathcal{K}$ = (everybody knows)$^\infty$ satisfies (K3) because for large enough N we have (everybody knows)^N = (everybody knows)$^{N+1}$. Therefore, $\mathcal{K} = \mathcal{K}^2 \subseteq \mathcal{K}^2$.

Exercise 12.12.23

Only the case of exercise 12.12.14 is considered. See figure 12.12.14 for the communal possibility sets. The event that both Beatrice and Carol have dirty faces is $D_{B,N} = \{7, 8\}$. If this event occurs, then either the true state is $\omega = 7$ or $\omega = 8$. Eventually, $\mathcal{M}(7) = \{7\}$ and $\mathcal{M}(8) = \{8\}$. In both cases, $\mathcal{M}(\omega) \subseteq D_{B,N}$.

Exercise 12.12.24

(a) All of Gino's possibility sets have three elements.

(b) Gino's announcement will not change Polly's possibility sets because Gino has added no information.

(c) Gino can now reason that if the true state were 9, then Polly would respond with 1. If she doesn't, then Gino knows the tre state is not 9. If any other state is the true state, Polly will respond with an answer of 4. Therefore, no further updating is possible.

(d) See figure 12.12.24.

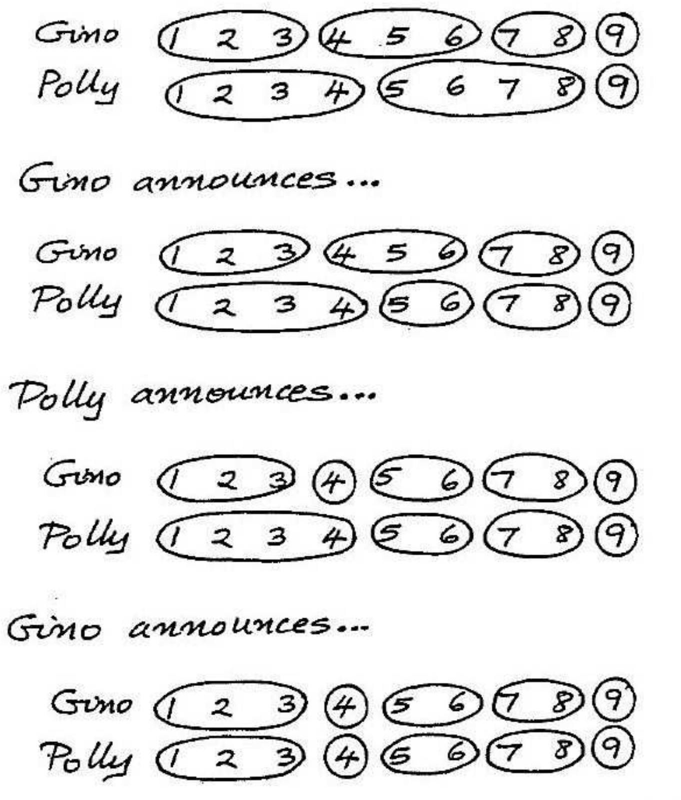


Figure 12.12.24

(e) This is true because $\mathcal{M}(5) = \{5, 6\} \subseteq E$. Yes, E is a common truism.

Exercise 12.12.25

(a) Gino realizes that either $\{1,2,3\}$ or $\{4,5,6\}$ are the only two possibility sets that contain an element of F. The possibility set $\{7,8,9\}$ has no elements in common with F. Using Bayes' rule he will determine that any of $\{1,2,3,4,5,6\}$ are equally likely.

(b) Polly's possibility partition is shown in figure 12.12.25(a). If the event F occurred, she would think only the states $\{1,2,3,4\}$ are possible. Therefore, she would assign a probability of zero to every other state. Again, she would use Bayes' rule to determine each of states 1, 2, 3, and 4 are equally likely. The event F contains half the elements in her possibility set containing F.

(c) Gino's possibility partition is shown in figure 12.12.25(b). He notices that Polly assigns all of the probability among the states in her first possibility set. Therefore, he can distinguish 4 from 5 and 6. Therefore, if the true state was 4 he would know that with certainty. If either 1, 2, or 3 occur, he will believe F occurred with probability $\frac{1}{3}$.

(d) Polly realizes that if the state is 4, Gino will know with certainty that F occurred. Therefore, she can distinguish between 4 and $\{1,2,3\}$. She still assigns the probability zero to F if any other state occurs. Her new possibility partition is shown in figure 12.12.25(c).

(e) The communal possibility partition is shown in figure 12.12.25(d). The players know that if $\omega \in \{1,2,3\}$ both will believe F occurred with probability $\frac{1}{3}$. If $\omega = 4$ then they know F occurred with certainty. Otherwise, they both know F did not occur.

(f) No player's posterior probability for F is common knowledge in figure 12.13(a) because the communal possibility partition is the set $\{1,2,3,4,5,6,7,8,9\}$.

(g) Gino's first announcement would be $\frac{1}{3}$ as in part (a). Polly would then announce that she believes F occurred with probability $\frac{1}{3}$ also because she knows states 5 or 6 are not the true state, so she knows 4 is not the true state by Gino's announcement. Both believe only states 1, 2, or 3 to be possible.

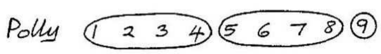


Figure 12.12.25(a)

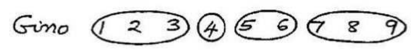


Figure 12.12.25(b)

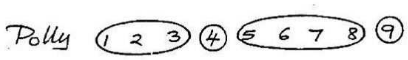


Figure 12.12.25(c)

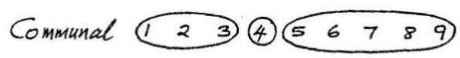


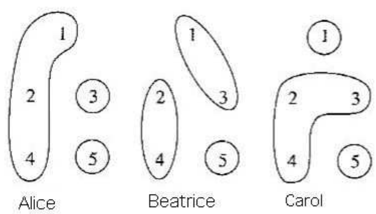
Figure 12.12.25(d)

Exercise 12.12.26

(a) See figure 12.12.26(a).

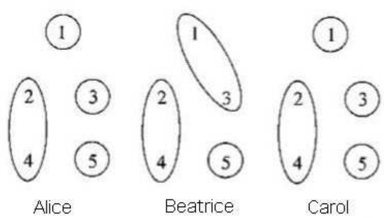
(b) See figure 12.12.26(b).

(c) The same as in figure 12.12.26(b).



State	Alice	Beatrice	Carol	Average
1	$\frac{2}{3}$	1	1	$\frac{8}{9}$
2	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{11}{18}$
3	1	1	$\frac{2}{3}$	$\frac{8}{9}$
4	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{11}{18}$
5	0	0	0	0

Figure 12.12.26(a)



State	Alice	Beatrice	Carol	Average
1	1	1	1	1
2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
3	1	1	1	1
4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
5	0	0	0	0

Figure 12.12.26(b)

(d) 2.

(e) See figure 12.12.26(c)

(f) No.

(g) After the first announcement.

(h) After the second announcement.

(i) Everybody's posterior probabilities will equal the average.

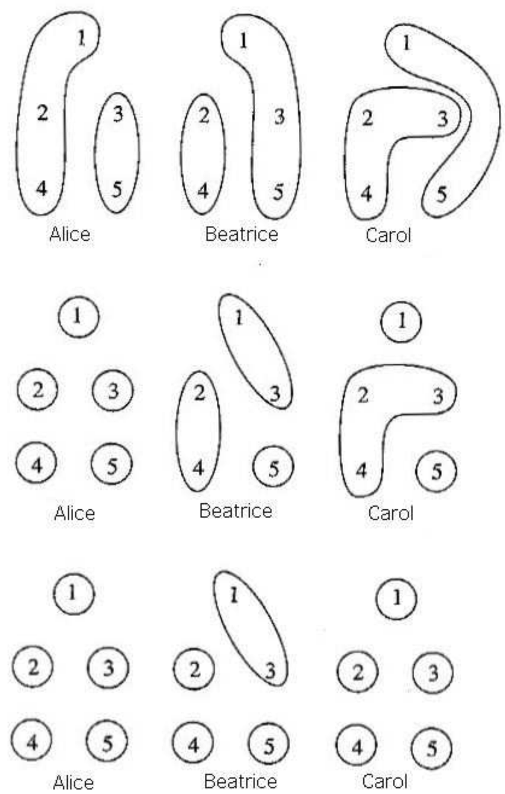


Figure 12.12.26(c)

Exercise 12.12.27

Bayesian-rational players optimize given their information. If each knows the strategy choice of the other, each will therefore make a best reply to the strategy chosen by the other.

Exercise 12.12.28

Whatever bet Alice proposes to Bob, Bob knows that Alice knows what cards she is holding. Hence, Alice will propose a bet if and only if she can make money of it. But then Bob will be losing money so he shouldn't accept any bets that Alice is willing to propose him. The same applies to Alice proposing Bob a bet over the possibility of time travel.