Exercise 11.9.1

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Exercise 11.9.2

The set $H$ has 4 elements. If $Z$ were a 3x4 matrix, $H$ would have 12 elements. If it was just before $Z$ was played the fifth time then $H$ would have $4^4$ elements. This is so because every time that $Z$ is played, there are 4 possible outcomes. So in total there are $4^4$ possible histories after 4 repetitions of $Z$. 
Exercise 11.9.3

The formal proof is by induction. Clearly, the statement is true for \( n = 1 \), since then this is just a regular Prisoners’ Dilemma. Now suppose that the statement is true for \( n - 1 \) repetitions. We want to prove that it is then also true for \( n \) repetitions. By the reasoning outlined in the previous exercise, there are \( 4^{n-1} \) possible histories in \( n - 1 \) repetitions of the Prisoners’ Dilemma. Thus, at the \( n \)-th stage a player has \( 2^{4^{n-1}} \) history-dependent choices for her move at that stage. Since her number of strategies in \( n \) repetitions is the number of strategies in \( n - 1 \) repetitions times the number of history dependent possible moves at the \( n \)-th stage, the formula is also true for \( n \) repetitions. This completes our formal proof by induction.
Exercise 11.9.4

Proof goes by induction on $n$. For $n = 1$ the statement is true since the $G$ has a unique Nash equilibrium, thus it also has a unique subgame-perfect equilibrium. Now suppose that the statement is true for $n - 1$ repetitions, and we need to prove that it then also holds for $n$ repetitions. In any subgame at the $n$-th stage players play the unique Nash equilibrium of the subgame which is the same for all subgames at the $n$-th stage. Thus the subgame-perfect equilibrium of $G^n$ will be the subgame-perfect equilibrium of the game which is obtained by adding the Nash equilibrium payoffs of $G$ to the terminal nodes in $G^{n-1}$. Call this game $G_0^{n-1}$. Since to obtain $G_0^{n-1}$ the same payoffs were added in all terminal nodes of $G^{n-1}$, the same strategies that form a subgame-perfect equilibrium in $G^{n-1}$ also form a subgame-perfect equilibrium in $G_0^{n-1}$. Thus, from the induction hypothesis $G_0^{n-1}$ has a unique subgame-perfect equilibrium, and consequently, $G^n$ has a unique subgame-perfect equilibrium in which the Nash equilibrium strategy is played at every stage.
Chapter 9

Exercise 11.9.5

Every equilibrium that is played in all the subgames of the last stage will induce 3 subgame-perfect equilibria of the game (since any of the 3 equilibria can be played when we backward induct to the first stage for any given play in the last stage). Thus, there are at least 9 subgame-perfect equilibria. Note that there may be more since different equilibria may be played in different subgames at the second stage.
Exercise 11.9.6

We prove this by contradiction. Suppose that in a node that is reached player 1 plays $D$, and assume that this is the last such node (where player 1 plans to play $D$). Then, regardless of what the other player is playing at that node, player 1 has a profitable deviation by playing $H$ at that node and $H$ at all nodes from then on. Thus, his original plan to play $D$ couldn’t possibly be an equilibrium play. A Nash equilibrium which is not subgame-perfect is for both players to play $H$ in the first stage, and play $H$ whenever the other has played $H$ every time before, and $D$ otherwise.
Chapter 9

Exercise 11.9.7

At every stage there is a unique stage subgame-perfect equilibrium. Now the argument is exactly the same as in Exercise 11.9.4, where “stage Nash equilibrium” is replaced by “stage subgame-perfect equilibrium”.

Exercise 11.9.8

Deviating from GRIM has to be unprofitable at any stage \( N \) in order for GRIM to constitute a Nash equilibrium. From Figure 1.3 we thus have that it has to be

\[
3(1-p)^N - ((1-p)^{N+1} + (1-p)^{N+2} + ...) \leq (1-p)^N 2((1-p)^N + (1-p)^{N+1} + (1-p)^{N+2} + ...)
\]

Since \((1-p)^{N+1} + (1-p)^{N+2} + ... = \frac{1-p}{p}\) and \((1-p)^N + (1-p)^{N+1} + (1-p)^{N+2} + ... = \frac{1}{p}\), this reduces, after canceling out the terms \((1-p)^N\) to \(3 - \frac{1-p}{p} \leq \frac{2}{p}\). Parameter \( p \) will be largest when this holds as an equality, so that \(3 - \frac{1-p}{p} = \frac{2}{p}\), which gives \( p = \frac{3}{4}\).
Exercise 11.9.9

Counting to 101 would mean having 101 states, thus the agents are never sure when the game will end because they can’t distinguish between game 100 and 101. This means that (GRIM, GRIM) can be a Nash equilibrium.
Exercise 11.9.10

The minimax payoff for Chicken of Figure 6.15(a) is (0, 0), the strategy being slow. See Figure 11.9.10(a) for the cooperative region and the region of possible payoffs given by the Folk Theorem. The minimax payoff for the Battle of Sexes of Figure 6.15(b) is \((\frac{2}{3}, \frac{2}{3})\), and the strategy is to play one's preferred event with probability equal to \(\frac{1}{3}\). See Figure 11.9.10(b).

![Figure 11.9.10(b)](image.png)

**Draw, scan, attach figures**
Exercise 11.9.11

The minimax payoffs are $(2, 2)$, and are given by playing *Hawk*, see Figure 11.9.11 for the sustainable payoffs of the repeated version of the game, as given by the Folk Theorem.

Figure 11.9.11
Exercise 11.9.12

The Moore machine determined by those specifications is TIT for TAT. This is quite straight-forward once you notice that the transition function only depends on what the opponent did in the previous period.
Exercise 11.9.13

Such computer can only distinguish between a finite number of states and it can also only implement a finite number of strategies, which only depend on finite histories. Thus, it is a finite automaton. The answer to the second part is “No”.
Chapter 9

Exercise 11.9.14

If the interest rate is \( r \) then the discount factor is \( \frac{1}{1+r} \), so that in our case the discount factor is \( \frac{1}{1.1} = 0.909 \). The price of the asset (if the first payment is at the beginning of the first year) is

\[
1000 \left( \frac{1}{1+r} + \left( \frac{1}{1+r} \right)^2 + \ldots \right) = 1000 \frac{1}{1 - \frac{1}{1+r}} = 1000 \frac{1+r}{r},
\]

which is equal to 11000 when \( r = 0.1 \).
Exercise 11.9.15

(a) Because the interest rate is cumulative - you don’t hold 1000 dollars for the whole year, so that in fact the yearly interest is much higher.

(b) The present value is

\[
1000 - 100 \frac{1}{1 + m} - 100 \left( \frac{1}{1 + m} \right)^2 - \ldots - 100 \left( \frac{1}{1 + m} \right)^{12} = 1000 - 100 \frac{1 + m - \frac{1}{(1 + m)^{12}}}{m},
\]

which is equal to 0 whenever approximately \(10m = 1 + m\), so that \(m = \frac{1}{9}\).

(c) The yearly interest rate is \(r = (1 + \mu)^{12} - 1\), which is equal to approximately 2.54, so that the yearly interest rate is more than 254 per cent!
Chapter 9

Exercise 11.9.16

The Folk Theorem for mixed strategies is the same as Theorem 11.2, except that the minimax point is the one obtained from mixed strategies (and is equal to maximin because of Von Neuman’s theorem), since we can now allow the choices to depend on randomization. Moreover, the set of possible outcomes with randomizations is the cooperative payoff region above the minimax point, rather than being dense in that set. It is important that each player directly observes the randomizing device of the opponent to be able to detect a deviation by the other player from the particular mixed equilibrium strategy (if randomizing devices were unobservable, a player could always blame it on a “bad draw” of chance and pretend he hadn’t deviated even if he had).
Exercise 11.9.17

A coin toss can serve as a coordinating device. In the Battle of Sexes, the players could coordinate on one outcome if “heads” come up and on another one if the result of the toss is “tails”.
Exercise 11.9.18

The problem is that nobody will punish Pandora if she deviates by taking the whole dollar - in the subgame where she has taken the whole dollar, she has no incentives to minimax herself, because also in the subgame where she hasn’t punished herself there is nobody to punish her, and so on ad infinitum.
Exercise 11.9.19

This strategy profile is both, a Nash equilibrium profile and a subgame-perfect equilibrium profile. The reason is that in the subgame where Alice has acquiesced in the past it is optimal for her to always acquiesce, given the behavior of the entrants. The same holds for the entrants. This is independent of the discount factor $\delta$. This shows that Alice’s reputation is an important factor in the infinitely repeated play.
Exercise 11.9.20

(a) Once Eve is faced with an unfair offer it is optimal for her to say “Yes”, so that is what she should do in that subgame. Backing up, it is optimal for Adam to make her an unfair offer.

(b) Pure one-shot Nash equilibria are (unfair, Yes) and (fair, No), and the mixed equilibria are (fair, p), where $p \in [0, \frac{2}{3}]$ is the probability with which Eve says “Yes”.

(c) The minimax point is (2, 1) so that the claim is true by the Folk Theorem 11.2.
Exercise 11.9.21

People would call the outcome \((2, 2)\) fair because the society has selected it as the fair equilibrium of the infinitely-repeated game. If made an unfair offer which they refused, people would normally make a reputation argument. The reason it is difficult to distinguish this from people having a taste for good reputation is because the behavior is impossible to distinguish.
Exercise 11.9.22

(a) In the subgame after the Queen’s choice, it is optimal for everyone to choose whatever she has chosen. So it is optimal for the Queen to choose “ball”.

(b) If it is common knowledge that Queen will play “ball” then it is optimal to play “ball”, since if somebody deviates by playing “box”, the others will still play “ball” (since if they believe everyone will play “ball” then the optimal action to play is “ball”) and everyone gets a payoff of 0.
Exercise 11.9.23

Whoever deviates is “not pure at heart” so he or she is punished by the rest. By punishing disidents the society is able to sustain anything as the truth.
Exercise 11.9.24

The strategies are to, whenever matched with someone who had previously deviated from cooperation, minimax that player. Otherwise cooperate. For $\delta$ sufficiently large, these strategies constitute an equilibrium profile.
Exercise 11.9.25

If a player (call him Bob) deviates then eventually everyone will have met a player who is punishing because she either met Bob, or she met someone who is punishing because of Bob (this explains the name contagion). Thus, eventually the whole society stops cooperating, so whomever Bob met, he would only be able to get the non-cooperative payoff from some moment on. If Bob is sufficiently patient, then the threat of such a bad outcome indeed will be enough to deter him from deviating, so that cooperation will be a subgame-perfect equilibrium.
Exercise 11.9.26

For “contagion” see the answer to the previous exercise. This is an instance where the end seems to justify the means, mainly because the means are purely counterfactual (the bad punishment never arises in equilibrium). When anyone from an outsider group is punished for somebody’s crime (that somebody is an outsider himself) this is called discrimination. After all, why should anyone at all be an outsider? Moreover, this will only deter the outsiders from committing crimes, and not the insiders.
Chapter 9

Exercise 11.9.27

Because mothers can’t punish their daughters for not cooperating.
Exercise 11.9.28

(a) If Alice deviates from $s$ at the $n$-th stage, where $n < 100$, then Bob will from then on always play “speed” as strategy $s$ instructs him to do (since at the $n$-th stage he will have played “slow” by $s$). Thus, Alice will get from the $n$-th stage on at most 0. Since

$$
\sum_{i=n}^{99} 2 + \frac{1}{4}2 + \frac{1}{4}0 + \frac{1}{4}3 - \frac{1}{4} \geq 3 + \sum_{i=n+1}^{100} 0,
$$

Alice has no incentives to deviate at any stage $n$ (a little care is required in the last stage but there she has no incentives to deviate because $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is a Nash equilibrium of the stage.

(b) The argument follows immediately from the answer to (a).

(c) Any income stream that gives 2 in penultimate stage and more than 1.5 on average in the other stages will work. For instance 3, 0, 3, 0, ..., 3, 0, 2, 1 works.

(d) Multiple Nash equilibria of the one-shot game allow for the punishment of the deviant player by resorting to a play of a one-shot Nash equilibrium which is bad for the deviant player.
Exercise 11.9.29

The proposed is not a Nash equilibrium. The mixed equilibrium of a stage of Battle of Sexes is $p = \frac{2}{3}$, $q = \frac{1}{3}$, where $p$ is the probability that player I plays “box”, and $q$ is the probability that player II plays “box”. Let $V(n)$ be the expected value of player I if both play $s$ in $n$-stage version of the BoS. The probability of each of the outcomes $(\text{box, box})$ and $(\text{ball, ball})$ in stage 1 is then $\frac{2}{9}$, and the probability that their actions don’t coincide is $\frac{5}{9}$. We can write the recursive formula for $V(n)$,

$$V(n) = \frac{2}{9}(2 + 1 + ...) + \frac{2}{9}(1 + 2 + ...) + \frac{5}{9}V(n - 1) = 3n\frac{2}{9} + \frac{5}{9}V(n - 1).$$

Now let $n$ be an odd number, and consider two possible deviations for I. Namely, if I instead plays box with certainty in stage 1, and then continues with $s$, his payoff is equal to

$$V_{\text{box}} = \frac{1}{3}(2 + 1 + ...) + \frac{2}{3}V(n - 1) = \frac{1}{3}(2 + 3\frac{n - 1}{2}) + \frac{2}{3}V(n - 1).$$

If I plays ball with certainty in stage 1, and then continues with $s$, his payoff is equal to

$$V_{\text{ball}} = \frac{2}{3}(1 + 2 + ...) + \frac{1}{3}V(n - 1) = \frac{2}{3}(1 + 3(n - 1)) + \frac{1}{3}V(n - 1).$$

Finally, let $V_p$ be I’s payoff from playing box with probability $p$ in stage 1, and $s$ from then on, keeping II’s strategy fixed at $s$. Clearly, $V_p$ is a linear function of $p$, so that I is either indifferent between all choices of $p$, or I strictly prefers either $V_{\text{box}}$ or $V_{\text{ball}}$. In particular, if $(s,s)$ is a Nash equilibrium, it must be that I is indifferent between $V_{\text{box}}$, $V_{\text{ball}}$ and $V(n) = V_\frac{3}{4}$. From $V_{\text{box}} = V_{\text{ball}}$ we have

$$\frac{2}{3} + \frac{n - 1}{2} + \frac{2}{3}V(n - 1) = \frac{2}{3} + n - 1 + \frac{1}{3}V(n - 1),$$

which implies that $V(n - 1) = \frac{3}{2}(n - 1)$. Inserting this into the recursive formula for $V(n)$ we obtain a contradiction, namely $\frac{3}{2}n = V(n) = 3n\frac{2}{9} + \frac{5}{9}V(n - 1) = \frac{3}{2}n - \frac{5}{6}$. 29