Exercise 10.8.1

Where the isoprofit curves touch the gradients of the profits of Alice and Bob point in the opposite directions. Thus, increasing one agent's profit will necessarily decrease the other's. The only candidates for Pareto-efficient outcomes are the outcomes where Alice and Bob share the monopoly profits in some way; this is the best that Alice and Bob can possibly do together, so their collusion should consist of splitting the monopoly profits and the monpoly production. The computations are as follos. Alice's profit is $\pi_1(a,b) = (K-c-a-b)\,a$, thus the gradient is $(K-c-b-2a, -a)$. Similarly, the gradient of Bob's profit is $(-b, K-c-2b-a)$. Thus, the Pareto-efficient points are those where $\frac{K-c-b-2a}{a} = \frac{K-c-2b-a}{b}$, which gives the equation $K-c = b+a$ for the Pareto-efficient production levels - this is precisely what we noted it should be. Since the Nash-equilibrium of the game is $a = b = \frac{K-c}{3}$ it is clearly not Pareto-efficient.

Exercise 10.8.2

(a) For Alice: $\pi_1(q_1, q_2) = q_1(K - q_1 - q_2 - c_1)$. Taking the derivative w.r.t. q_1 we obtain $\frac{\partial \pi_1}{\partial q_1} = K - 2q_1 - q_2 - c_1$, and setting it equal to 0 we get Alice's reaction curve. Similarly for Bob.

(b) To obtain the Nash equilibrium, we need to solve the system of equations $2q_1 = K - q_2 - c_1$ and $2q_2 = K - q_1 - c_2$. The solution of this system is $q_1 = \frac{1}{3}(K + c_2 - 2c_1)$, $q_2 = \frac{1}{3}(K + c_1 - 2c_2)$.

(c) Plug the solution from part (b) into the profit of each agent.

Exercise 10.8.3

(a) and (b) See Fig10_7_3.nb.

(c) As in exercise 10.8.1. the Pareto-efficient outputs are obtained from the gradient condition which is now $\frac{K-2q_1-q_2-c_1}{q_1} = \frac{K-2q_2-q_1-c_2}{q_2}$. After some manipulation, this yields the desired curve. Setting either $q_2 = 0$ or $q_1 = 0$ we obtain the Monopoly profits for either agent Bob or Alice. It is easily checked that the Nash equilibrium outputs from part (b) don't lie on the curve (just plug them in).

Figure missing

Exercise 10.8.4

(a) $\pi_1(q_1, q_2) = (K - q_1 - q_2 - c_1) q_1$. Differentiating, we obtain $\frac{\partial \pi_1}{\partial q_1} = K - 2q_1 - q_2 - c_1$, so that $q_1^*(q_2) = \frac{K - q_2 - c_1}{2}$. Similarly for Bob.

(b) To obtain the equilibrium solve the system of equations $q_1^* = \frac{K - q_2^* - c_1}{2}$ and $q_2^* = \frac{K - q_1^* - c_2}{2}$ to get $q_1^* =$

Exercise 10.8.5

(a) Player I's reaction curve is $q_1^*(q_2) = \frac{1}{4}(M - c_1 + q_2)$, and player II's is $q_2^*(q_1) = \frac{1}{4}(M - c_2 + q_1)$.

(b) The equilibrium quantities are $q_1^* = \frac{1}{15}(5M - 4c_1 - c_2)$ and $q_2^* = \frac{1}{15}(5M - 4c_2 - c_1)$. The equilibrium prices are $p_1^* = \frac{1}{15}(10M + 7c_1 - 2c_2)$ and $p_2^* = \frac{1}{15}(10M + 7c_2 - 2c_1)$.

(c) The equilibrium profits are $\pi_1^* = \frac{1}{225}(5M - 4c_1 - c_2)(10M - 8c_1 - 2c_2)$ and $\pi_2^* = \frac{1}{225}(5M - 4c_2 - c_1)(10M - 8c_2 - 2c_1)$. In exercise 10.8.4 the two goods are substitutes, and in this exercise the goods are complements.

Exercise 10.8.6

(a) The fixed cost of F would not deter entry as long as F is less than the equilibrium profit, because the firms would still be better off entering and earning their equilibrium profit minus F, than they would by not entering and earning zero.

(b) We know from section 10.2 that a firm's profit in the n-player oligopoly game will be $\left[\frac{1}{n+1}(M-c)\right]^2$. Therefore, the condition which determines the number of firms in the industry is $\left[\frac{1}{n+1}(M-c)\right]^2 \geq F$. With a little algebra, this reduces to $n \leq F^{-\frac{1}{2}}(M-c)$. As $F \to 0$, the number of firms in the industry will become arbitrarily large.

Exercise 10.8.7

Suppose Bob offered a price $p > c_1$. Then Alice could undercut him, and offer a price $\tilde{p}, p > \tilde{p} > c_1$, to capture the whole market, hence this would not be an equilibrium. On the other hand if Bob offered a price $p < c_1$, then even by increasing his price a tiny bit to $\tilde{p}$, $p < \tilde{p} < c_1$, Bob would still keep the whole market, and make a bigger profit. The only possible equilibrium is then for Alice and Bob to both offer $p = c_1$. In this equilibrium every customer in the market needs to be buying from Bob. The reason is that if Alice offered a higher price than Bob, then Bob could also slightly increase his price and still capture the whole market, and we would be in the first case above. If Alice offered a lower price then she would be making a negative profit so she would be better off not getting any business. Finally, if they offered the same prices but not everyone bought from Bob, then Bob could slightly undercut Alice and capture the whole market for nearly the same price (and make a higher profit). Indeed, everyone needs to buy from Bob.

Exercise 10.8.8

In exercise 10.8.4 begin by solving the demand equations for q_1 and q_2 in terms of p_1 and p_2. This yields $q_1 = \frac{1}{3}(M+p_1-2p_2)$ and $q_2 = \frac{1}{3}(M+p_2-2p_1)$. It is then possible to express the players; profits in terms of p_1 and p_2. Next compute $\frac{\partial \pi_i}{\partial p_i}$. The best reply correspondences $p_1 = R_1(p_2) = \frac{1}{4}(M+p_2+2c_1)$ and $p_2 = R_2(p_1) = \frac{1}{4}(M+p_1+2c_2)$ are found by setting each partial derivative equal to zero. An equilibrium occurs when both the best reply equations hold simultaneously. The unique Nash equilibrium is therefore $(\tilde{p}_1, \tilde{p}_2)$, where $15\tilde{p}_1 = 5M + 8c_1 + 2c_2$ and $15\tilde{p}_2 = 5M + 8c_2 + 2c_1$.

In exercise 10.8.5 matters are much the same except that the best reply correspondences are $p_1 = R_1(p_2) = \frac{1}{3}(3M-p_1-2p_2)$ and $p_2 = R_2(p_1) = \frac{1}{3}(3M-2p_1-p_2)$.

Exercise 10.8.9

(a) Figure 10.8.9 illustrates the situation. A customer who is just indifferent between buying from player I and buying from player II pays $p + ty^2 = P + tY^2$, where $l = x + y + Y + X$. It is worth noting that

$$p_P = t(Y^2 - y^2) = t(Y - y)(Y + y) = tz(Y - y),$$

where $z = l - x - X$. We can deduce that, for fixed x and X, $\frac{\partial y}{\partial p} = \frac{\partial Y}{\partial P} = -\frac{1}{2}tz$. Player I gets $\rho(x + y)$ customers, and so his profit is $\pi_1 = \rho(p - c)(x + y)$. Similarly, $\pi_2 = \rho(P - c)(X + Y)$.

(b) To find player I's best-reply correspondence, we compute

$$\frac{\partial \pi_1}{\partial p} = \rho(x + y) + \rho(p - c)\frac{\partial y}{\partial p}.$$

Setting this equal to zero leads to $p = 2tz(x+y)+c$. Doing the same thing for player II yields $P = 2tz(X+Y)+c$. Use these equations to obtain formulas for $p - P$ and $p + P$. One can then substitute for the expressions $y - Y$ and $y + Y$ in these formulas using the results of part (a). This leads to the conclusion that player I's equilibrium price is

$$p = \frac{1}{3}tz(x - X + 3l) + c.$$

His equilibrium profit is therefore

$$\pi_1 = \rho(p - c)(x + y) = \frac{\rho}{2tz}(p - c)^2 = \frac{1}{18}\rho t(l - x - X)(x - X + 3l)^2.$$

Notice that if they both locate at the same place, then price competition drives the profits to zero.

(c) In the location game the players choose x and X, assuming that their profits will be as calculated in part (b). The sign of $\frac{\partial \pi_1}{\partial x}$ is the same as that of

$$-(x - X + 3l)^2 + 2(l - x - X)(x - X + 3l) = -(x - X + 3l)(l + 3x + X),$$

and thus is always negative when $0 \le x \le l$ and $0 \le X \le l$. It follows that player I's profit is maximized by his locating as far from player II as he can . The same is also true of player II. In equilibrium, the players therefore locate at the two ends of the street.

(d) We have calculated strategies for the whole game that induce Nash equilibrium behavior in each price-fixing subgame. We have therefore found that the whole game has a unique subgame-perfect equilibrium.

(e) The firms will locate at the ends of the street so that $x = X = 0$ and $z = l$. The equilibrium prices are $p = P = c + \frac{1}{3}tl^2$. The equilibrium profits are $\pi_1 = \pi_2 = \frac{1}{2}\rho t l^2$.

***** I think it is wrong. Needs to be checked*****

Must attach figure: figure 7.9.4 in Linster, p62

Exercise 10.8.10

In this game, player I will produce $q_1 = \frac{M-c}{2}$, player II will produce $q_2 = \frac{M-c-q_1}{2}$. Player m will produce $q_m = \frac{M-c-\sum_{i<m} q_i}{2}$. The vector of outputs will be $(\frac{1}{2}(M-c), \frac{1}{4}(M-c), \frac{1}{8}(M-c), \ldots, \frac{1}{2^n}(M-c))$. If we sum these we get $(1-\frac{1}{2^n})(M-c)$. As $n \to \infty$, this approaches $(M-c)$, which is the competitive outcome.

Exercise 10.8.11

If player I chooses first, he will produce $q_1 = \frac{M-c}{2}$. After seeing his output choice, everyone else will produce $q_i(q_1) = \frac{1}{n}(M - c - q_1)$, where n is the total number of firms. Total output is $\sum_{j=1}^{n} q_j = \frac{1}{2}(M-c) + \frac{n-1}{2n}(M-c)$. As $n \to \infty$, this approaches the competitive outcome.

Exercise 10.8.12

It is only necessary to observe from the answer to exercise 10.8.9(a) that it is best for player I to choose $x = 0$ whatever he may know about X.

Exercise 10.8.13

Let $F(p)$ be the probability that the opposing player posts a price higher than p. The profit function of a player if he posts a price p is then

$$\Pi(p \mid F) = F(p)p^{-l}(p - c).$$

Substituting $F(p) = \left(\frac{a-c}{p-c}\right)\left(\frac{p}{a}\right)^l$, we get that $\Pi(p \mid F) = \frac{a-c}{a}$. This implies that given such distribution over prices by the opposing player, each player is indifferent between posting any price, hence this constitutes a mixed-strategy Nash equilibrium of the Bertrand game.

Exercise 10.8.14

It would be a strictly dominated strategy for a player to post a price higher than p^*, since he could instead post a price p^* and thus (weakly) increase the probability of capturing the market and strictly increase his profit conditional on capturing the market.

Again let $F(p)$ be the probability that the opposing player posts a price higher than p in a symmetric mixed Nash equilibrium with support $[a, b]$ (suppose such F existed). By definition, it has to be that $F(a) = 1$ and $F(b) = 0$, and since it is a Nash equilibrium, each player has to be indifferent between all the prices, given F. By this indifference condition, it has to be that the profit of a player, $\Pi(p \mid F)$, is constant with respect to p. Since $\Pi(p \mid F) = F(p)p^{-l}(p-c)$ this implies that $F(p) = K\frac{p^l}{p-c}$. Inserting $F(a) = 1$, we get that $K = \frac{a-c}{a^l}$, but then $F(b) > 1$, which is a contradiction.

Exercise 10.8.15

Let $F(p)$ be the probability that the opposing player posts a price higher than p. We need to construct an F, such that a player gets at least as much by randomizing according to F, as with any other strategy minus ϵ, if the opposing player randomizes according to F. For this to be true, it is good enough to construct a mixed equilibrium in which a player gets at most ϵ from any of his pure strategies, if the opposing player randomizes according to F. A player's profit from posting a price p is then $\Pi(p \mid F) = F(p)p^{-l}(p-c)$. First note that $F(b) = 0$ so that $\Pi(0 \mid F) = 0$. Next, we set a such that $\Pi(a \mid F) < \epsilon$. Since $F(a) = 1$, $\Pi(a \mid F) = a^{-l}(a-c)$, so that setting $a = \epsilon c^l + c$ we have $\Pi(a \mid F) = \epsilon \left(\frac{c}{a}\right)^l < \epsilon$. Now let $F(p) = \left(\frac{b-p}{b-a}\right)^k$, and we will set k such that $\Pi'(p \mid F) < 0$, $\forall p \in [a,b)$. With this form of F, we get that, $\Pi'(p \mid F) = (b-a)^{-k}(b-p)^{k-1}p^{-l-1}\left[(b-p)p - l(p-c)(b-p) - kp(p-c)\right]$, and this negative whenever $(b-p)p - l(p-c)(b-p) - kp(p-c) < 0$. For this last expression to be negative for all $p \in (a,b)$, it is enough to set k large enough that this expression is negative at, a. In particular, setting $k_\epsilon = \frac{b-a}{a-c} = \frac{b-a}{\epsilon c^l}$ we obtain $(b-a)a - l(a-c)(b-a) - k_\epsilon a(a-c) = -l(a-c)(b-a) < 0$.

Now we show that this F describes an ϵ-equilibrium. First note that by above, $\Pi(p \mid F) < \epsilon$ for every price $p \in [a,b)$. Hence the profit to each player from randomizing according to F is less then ϵ. Since $\Pi(p \mid F) = p^{-l}(p-c) < a^{-l}(a-c) = \Pi(a \mid F)$, for every $p \in [a,c)$. Thus, if the opponent randomizes according to F, the profit to a player from any pure strategy is less than ϵ, hence, the expected profit from any mixed strategy F' is less than ϵ, which shows that (F,F) is an ϵ equilibrium.

The probability density function in this mixed equilibrium puts larger and larger weight on points close to a, as ϵ tends to 0. As ϵ tends 0, k_ϵ in the above F tends to ∞, and a tends to c, so that in the limit both players put all the probability on $p = c$ which is the pure-strategy Nash equilibrium of the Bertrand pricing game. In particular, F tends point-wise to the pure strategy $p = c$. As $\epsilon \to 0$, players' payoffs tend to 0.