COURSE SUMMARY

This chapter is a brief review of engineering economic analysis/engineering economy. The goal is to give you a better grasp of the major topics in a typical first course. Hopefully, this overview will help you put the course lectures and your reading of the textbook in better perspective. There are 28 example problems scattered throughout the engineering economics review. These examples are an integral part of the review and should be worked to completion as you come to them.

CASH FLOW

The field of engineering economics uses mathematical and economic techniques to systematically analyze situations that pose alternative courses of action.

The initial step in engineering economics problems is to resolve a situation, or each alternative course in a given situation, into its favorable and unfavorable consequences or factors. These are then measured in some common unit -- usually money. Those factors that cannot readily be reduced to money are called intangible, or irreducible, factors. Intangible or irreducible factors are not included in any monetary analysis but are considered in conjunction with such an analysis when making the final decision on proposed courses of action.

A cash flow table shows the “money consequences” of a situation and its timing. For example, a simple problem might be to list the year-by-year consequences of purchasing and owning a used car:

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beginning of first Year 0</td>
<td>-4500</td>
<td>Car purchased “now” for $4500 cash. The minus sign indicates a disbursement.</td>
</tr>
<tr>
<td>End of Year 1</td>
<td>-350</td>
<td>Maintenance costs are $350 per year</td>
</tr>
<tr>
<td>End of Year 2</td>
<td>-350</td>
<td></td>
</tr>
<tr>
<td>End of Year 3</td>
<td>-350</td>
<td></td>
</tr>
<tr>
<td>End of Year 4</td>
<td>+2000</td>
<td>The car is sold at the end of the 4th year for $2000. The plus sign represents a receipt of money.</td>
</tr>
</tbody>
</table>

This same cash flow may be represented graphically:
The upward arrow represents a receipt of money, and the downward arrows represent disbursements.

The x-axis represents the passage of time.

**EXAMPLE 1**


**Solution**

Unless otherwise stated in problems, the customary assumption is a beginning-of-year purchase, followed by end-of-year receipts or disbursements, and an end-of-year resale or salvage value. Thus the plotter repairs and the plotter sale are assumed to occur at the end of the year. Letting a minus sign represent a disbursement of money, and a plus sign a receipt of money, we are able to set up this cash flow table:

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beginning</td>
<td>-$500</td>
</tr>
<tr>
<td>End of 2001</td>
<td>0</td>
</tr>
<tr>
<td>End of 2002</td>
<td>0</td>
</tr>
<tr>
<td>End of 2003</td>
<td>-85</td>
</tr>
<tr>
<td>End of 2004</td>
<td>+130</td>
</tr>
<tr>
<td>End of 2005</td>
<td>+160</td>
</tr>
</tbody>
</table>

Notice that at the end of 2005, the cash flow table shows +160. This is the net of -140 and +300.

If we define Year 0 as the beginning of 2000, the cash flow table becomes:

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$500</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-85</td>
</tr>
</tbody>
</table>
From this cash flow table, the definitions of Year 0 and Year 1 become clear. Year 0 is defined as the beginning of Year 1. Year 1 is the end of Year 1. Year 2 is the end of Year 2, and so forth.

TIME VALUE OF MONEY

When the money consequences of an alternative occur in a short period of time -- say, less than one year -- we might simply add up the various sums of money and obtain the net result. But we cannot treat money this same way over longer periods of time. This is because money today does not have the same value as money at some future time.

Consider this question: Which would you prefer, $100 today or the guarantee of receiving $100 a year from now? Clearly, you would prefer the $100 today. If you had the money today, rather than a year from now, you could use it for the year. And if you had no use for it, you could lend it to someone who would pay interest for the privilege of using your money for the year.

EQUIVALENCE

In the preceding section we saw that money at different points in time (for example, $100 today or $100 one year from today) may be equal in the sense that they both are $100, but $100 a year from today is not an acceptable substitute for $100 today. When we have acceptable substitutes, we say they are equivalent to each other. Thus at 8% interest, $108 a year from today is equivalent to $100 today.

EXAMPLE 2
At a 10% per year interest rate, $500 today is equivalent to how much three years from today?

Solution

$500 now will increase by 10% in each of the three years.

\[
\begin{array}{l}
\text{Now} = & \$500.00 \\
\text{End of 1st year} = & 500 + 10\%(500) = 550.00 \\
\text{End of 2nd year} = & 550 + 10\%(550) = 605.00 \\
\text{End of 3rd year} = & 605 + 10\%(605) = 665.50
\end{array}
\]

Thus $500 now is equivalent to $665.50 at the end of three years.

Equivalence is an essential factor in engineering economic analysis. Suppose we wish to select the better of two alternatives. First, we must compute their cash flows. An example would be:

<table>
<thead>
<tr>
<th>Year</th>
<th>Alternative A</th>
<th>Alternative B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$2000</td>
<td>-$2800</td>
</tr>
<tr>
<td>1</td>
<td>+800</td>
<td>+1100</td>
</tr>
<tr>
<td>2</td>
<td>+800</td>
<td>+1100</td>
</tr>
<tr>
<td>3</td>
<td>+800</td>
<td>+1100</td>
</tr>
</tbody>
</table>
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The larger investment in Alternative B results in larger subsequent benefits, but we have no direct way of knowing if Alternative B is better than Alternative A. Therefore we do not know which alternative should be selected. To make a decision we must resolve the alternatives into equivalent sums so they may be compared accurately and a decision made.

COMPOUND INTEREST FACTORS

To facilitate equivalence computations a series of compound interest factors will be derived and their use illustrated.

Symbols

\[ i = \text{Interest rate per interest period. In equations the interest rate is stated as a decimal (that is, 8\% interest is 0.08)}. \]
\[ n = \text{Number of interest periods}. \]
\[ P = \text{A present sum of money}. \]
\[ F = \text{A future sum of money. The future sum F is an amount, n interest periods from the present, that is equivalent to P with interest rate i}. \]
\[ A = \text{An end-of-period cash receipt or disbursement in a uniform series continuing for n periods, the entire series equivalent to P or F at interest rate i}. \]
\[ G = \text{Uniform period-by-period increase in cash flows; the arithmetic gradient}. \]
\[ g = \text{Uniform rate of period-by-period increase in cash flows; the geometric gradient}. \]

Functional Notation

<table>
<thead>
<tr>
<th>To Find</th>
<th>Given</th>
<th>Functional Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single Payment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Compound Amount Factor</td>
<td>F</td>
<td>P</td>
</tr>
<tr>
<td>Present Worth Factor</td>
<td>P</td>
<td>F</td>
</tr>
<tr>
<td>Uniform Payment Series</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sinking Fund Factor</td>
<td>A</td>
<td>F</td>
</tr>
</tbody>
</table>
## Course Summary

<table>
<thead>
<tr>
<th>Capital Recovery Factor</th>
<th>( A )</th>
<th>( P )</th>
<th>((A/P, i, n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compound Amount Factor</td>
<td>( F )</td>
<td>( A )</td>
<td>((F/A, i, n))</td>
</tr>
<tr>
<td>Present Worth Factor</td>
<td>( P )</td>
<td>( A )</td>
<td>((P/A, i, n))</td>
</tr>
</tbody>
</table>

**Arithmetic Gradient**

<table>
<thead>
<tr>
<th>Gradient Uniform Series</th>
<th>( A )</th>
<th>( G )</th>
<th>((A/G, i, n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient Present Worth</td>
<td>( P )</td>
<td>( G )</td>
<td>((P/G, i, n))</td>
</tr>
</tbody>
</table>

From the table above we can see that the functional notation scheme is based on writing (To Find / Given, \( i, n \)). Thus, if we wished to find the future sum \( F \), given a uniform series of receipts \( A \), the proper compound interest factor to use would be \((F/A, i, n)\).

### Single Payment Formulas

Suppose a present sum of money \( P \) is invested for one year at interest rate \( i \). At the end of the year, we receive back our initial investment \( P \) together with interest equal to \( Pi \) or a total amount \( P + Pi \). Factoring \( P \), the sum at the end of one year is \( P(1 + i) \). If we agree to let our investment remain for subsequent years, the progression is as follows:

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount at Beginning of Period</th>
<th>Interest for the Period</th>
<th>Amount at End of the Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st year</td>
<td>( P )</td>
<td>( Pi )</td>
<td>( P(1 + i) )</td>
</tr>
<tr>
<td>2nd year</td>
<td>( P(1 + i) )</td>
<td>( Pi(1 + i) )</td>
<td>( P(1 + i)^2 )</td>
</tr>
<tr>
<td>3rd year</td>
<td>( P(1 + i)^2 )</td>
<td>( Pi(1 + i)^2 )</td>
<td>( P(1 + i)^3 )</td>
</tr>
<tr>
<td>nth year</td>
<td>( P(1 + i)^{n-1} )</td>
<td>( Pi(1 + i)^{n-1} )</td>
<td>( P(1 + i)^n )</td>
</tr>
</tbody>
</table>
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The present sum $P$ increases in $n$ periods to $P(1 + i)^n$. This gives us a relationship between a present sum $P$ and its equivalent future sum $F$:
\[
F = P(1 + i)^n
\]
This is the Single Payment Compound Amount formula. In functional notation it is written
\[
F = P(F/P, i, n)
\]
The relationship may be rewritten as
\[
P = F(1 + i)^n
\]
This is the Single Payment Present Worth formula. It is written
\[
P = F(P/F, i, n)
\]
EXAMPLE 3
At a 10% per year interest rate, $500 today is equivalent to how much three years from today?

Solution
This problem was solved in Example 2. Now it can be solved using a single payment formula.
\[
P = $500 \quad F = \text{unknown} \\
n = 3 \text{ years} \quad i = 10\%
\]
\[
F = P(1 + i)^n = 500(1 + 0.10)^3 = $665.50
\]
This problem may also be solved using the Compound Interest Tables.
\[
F = P(F/P, i, n) = 500(F/P, 10\%, 3)
\]
From the 10% Compound Interest Table, read $(F/P, 10\%, 3) = 1.331$.
\[
F = 500(F/P, 10\%, 3) = 500(1.331) = $665.50
\]
EXAMPLE 4
To raise money for a new business, a friend asks you to loan her some money. She offers to pay you $3000 at the end of four years. How much should you give her now if you want to earn 12% interest per year on your money?
Solution

\[ F = \$3000 \quad P = \text{unknown} \]

\[ n = 4 \text{ years} \quad i = 12\% \]

\[ P = F(1 + i)^n = 3000(1 + 0.12)^4 = \$1906.55 \]

Alternate computation using Compound Interest Tables:

\[ P = F(P/F, i, n) = 3000(P/F, 12\%, 4) \]

\[ = 3000(0.6355) \]

\[ = \$1906.50 \]

Note that the solution based on the Compound Interest Table is slightly different from the exact solution using a hand calculator. In economic analysis, the Compound Interest Tables are always considered to be sufficiently accurate.

Uniform Payment Series Formulas

A uniform series is identical to \( n \) single payments, where each single payment is the same and there is one payment at the end of each period. Thus, the present worth of a uniform series is derived algebraically by summing \( n \) single-pa\( y\)nts. The derivation of the equation for the present worth of a uniform series is shown below.

\[ P = A \left[ \frac{1}{(1 + i)^1 + 1/(1 + i)^2 + \cdots + 1/(1 + i)^n} \right] \]

\[ (1 + i)P = A \left[ \frac{1}{(1 + i)^0 + 1/(1 + i)^1 + 1/(1 + i)^2 + \cdots + 1/(1 + i)^n} \right] \]

\[ (1 + i)P - P = A \left[ \frac{1}{(1 + i)^0} - 1/(1 + i)^n \right] \]

\[ iP = A \left[ 1 - 1/(1 + i)^n \right] \]

\[ = A \left[ (1 + i)^n - 1 \right]/(1 + i)^n \]

\[ P = A \left[ (1 + i)^n - 1 \right]/[i(1 + i)^n] \quad \text{Uniform Series Present worth formula} \]

Solving this equation for \( A \):

\[ A = P \left[ i(1 + i)^n \right]/[(1 + i)^n - 1] \quad \text{Uniform Series Capital Recovery formula} \]

Since \( F = P(1 + i)^n \), we can multiply both sides of the \( P/A \) equation by \((1 + i)^n \) to obtain

\[ (1 + i)^nP = A \left[ (1 + i)^n - 1 \right]/i \quad \text{which yields} \]

\[ F = A \left[ (1 + i)^n - 1 \right]/i \quad \text{Uniform Series Compound Amount formula} \]

Solving this equation for \( A \), we obtain
Course Summary

\[ A = F \frac{i}{(1 + i)^n - 1} \] Uniform Series Sinking Fund formula

In functional notation, the uniform series factors are

- Compound Amount(\(F/A, i, n\))
- Sinking Fund(\(A/F, i, n\))
- Capital Recovery(\(A/P, i, n\))
- Present Worth(\(P/A, i, n\))

**EXAMPLE 5**

If $100 is deposited at the end of each year in a savings account that pays 6% interest per year, how much will be in the account at the end of five years?

**Solution**

\[
\begin{align*}
A &= \$100 \\
F &= \text{unknown} \\
n &= 5 \text{ years} \\
i &= 6\% \\
F &= A(F/A, i, n) = 100(F/A, 6\%, 5) = 100(5.637) = \$563.70
\end{align*}
\]

**EXAMPLE 6**

A woman wishes to make a quarterly deposit into her savings account so that at the end of 10 years the account balance will be $10,000. If the account earns 6% annual interest, compounded quarterly, how much should she deposit each quarter?

**Solution**

\[
\begin{align*}
F &= \$10,000 \\
A &= \text{unknown} \\
n &= 40 \text{ quarterly deposits} \\
i &= 1\% \text{ per quarter year} \\
A &= F(A/F, i, n) = 10,000(A/F, 1\%, 40) = 10,000(0.0184) = \$184
\end{align*}
\]

**EXAMPLE 7**

An individual is considering the purchase of a used automobile. The total price is $6200 with $1240 as a down payment and the balance to be paid in 48 equal monthly payments with interest of 12% compounded monthly. The payments are due at the end of each month. Compute the monthly payment.

**Solution**

The amount to be repaid by the 48 monthly payments is the cost of the automobile minus the $1240 down payment.

\[
\begin{align*}
P &= \$4960 \\
A &= \text{unknown} \\
n &= 48 \text{ monthly payments} \\
i &= 1\% \text{ per month} \\
A &= P(A/P, i, n) = 4960(A/P, 1\%, 48) = 4960(0.0263) = \$130.45
\end{align*}
\]

**EXAMPLE 8**
A couple sold their home. In addition to a cash down payment, they financed the remaining balance for the buyer. The loan will be paid off by monthly payments of $232.50 for 10 years. The couple decides to sell the loan to a local bank. The bank will buy the loan, based on 1% per month interest. How much will the bank pay the couple for the loan?

**Solution**

\[ A = 232.50 \quad \text{P = unknown} \]
\[ n = 120 \text{ months} \quad i = 1\% \text{ per month} \]

\[ P = A(P/A, i, n) = 232.50(P/A, 1\%, 120) = 232.50(69.701) = 16,205.48 \]

---

**Arithmetic Gradient**

At times one will encounter a situation where the cash flow series is not a constant amount \( A \). Instead it is an increasing series like:

![Arithmetic Gradient Diagram](image)

This cash flow may be resolved into two components:

\[ P = P + G \]

We can compute the value of \( P^* \) as equal to \( P \) plus \( P \). We already have an equation for \( P^* = A(P/A, i, n) \)

The value for \( P \) in the right-hand diagram is
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\[ P = G \frac{(1+i)^n - in - 1}{i^2 (1+i)^n} \]

This is the Arithmetic Gradient Present Worth formula. In functional notation, the relationship is

\[ P = G(P/G, i, n) \]

EXAMPLE 9
The maintenance on a machine is expected to be $155 at the end of the first year and then increase $35 each year for the next seven years. What sum of money would need to be set aside now to pay the maintenance for the eight-year period? Assume 6% interest.

Solution

\[ P = 155(\frac{P}{A}, 6\%, 8) + 35(\frac{P}{G}, 6\%, 8) = 155(6.210) + 35(19.841) = $1656.99 \]

In the gradient series, if instead of the present sum \( P \), an equivalent uniform series \( A \) is desired, the problem becomes

\[ A^* = A + A' \]

The relationship between \( A' \) and \( G \) in the right-hand diagram is

\[ A' = G \frac{(1+i)^n - in - 1}{i(1+i)^n - 1} \]

In functional notation, the Arithmetic Gradient (to) Uniform Series factor is
$A = G(A/G, i, n)$

It is important to note carefully the diagrams for the two arithmetic gradient series factors. In both cases the first term in the arithmetic gradient series is zero and the last term is $(n - 1)G$. But we use $n$ in the equations and functional notation. The formula derivations were done on this basis and the arithmetic gradient series Compound Interest Tables are computed this way.

**EXAMPLE 10**
For the situation in Example 9, we wish now to know the uniform annual equivalent maintenance cost. Compute an equivalent $A$ for the maintenance costs.

**Solution**

![Diagram of geometric gradient]

Equivalent uniform annual maintenance cost:

$A = 155 + 35(A/G, 6\%, 8) = 155 + 35(3.195) = \$266.83$

**Geometric Gradient**

The arithmetic gradient is applicable where the period-by-period change in the cash flow is a uniform amount. There are other situations where the period-by-period change is a uniform rate, $g$. A diagram of this situation is

where $A_n = A_1(1 + g)^{n-1}$
$g$ = Uniform rate of period-by-period change; the geometric gradient stated as a decimal (8% = 0.08).
$A_1$ = Value of $A$ at Year 1.
$A_n$ = Value of $A$ at any Year $n$. 
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Geometric Series Present Worth Formulas:

When \( i = g \), \( P = A_t(n(1 + i)^{-1}) \)

When \( i \neq g \), \( P = A_t \left( \frac{1 - (1 + g)^n(1 + i)^{-n}}{i - g} \right) \)

EXAMPLE 11

It is likely that airplane tickets will increase 8% in each of the next four years. The cost of a plane ticket at the end of the first year will be $180. How much money would need to be placed in a savings account now to have money to pay a student’s travel home at the end of each year for the next four years? Assume the savings account pays 5% annual interest.

Solution

The problem describes a geometric gradient where \( g = 8\% \) and \( i = 5\% \).

\[
P = A_t \left( \frac{1 - (1 + g)^n(1 + i)^{-n}}{i - g} \right)
\]

\[
P = 180.00 \left( \frac{1 - (1.08)^4(1.05)^{-4}}{0.05 - 0.08} \right) = 180.00 \left( \frac{-0.119278}{-0.03} \right) = \$715.67
\]

Thus, \$715.67 would need to be deposited now.

As a check, the problem can be solved without using the geometric gradient:

<table>
<thead>
<tr>
<th>Year</th>
<th>( A_t )</th>
<th>Ticket</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$180.00</td>
<td>194.40</td>
</tr>
<tr>
<td>2</td>
<td>$194.40</td>
<td>209.95</td>
</tr>
<tr>
<td>3</td>
<td>$209.95</td>
<td>226.75</td>
</tr>
</tbody>
</table>

\[
P = 180.00(P/F, 5\%, 1) + 194.40(P/F, 5\%, 2) + 209.95(P/F, 5\%, 3) + 226.75(P/F, 5\%, 4)
\]

\[
= 180.00(0.9524) + 194.40(0.9070) + 209.95(0.8638) + 226.75(0.8227)
\]

\[
= \$715.66
\]

NOMINAL AND EFFECTIVE INTEREST

Nominal interest is the annual interest rate without considering the effect of any compounding.

Effective interest is the annual interest rate taking into account the effect of any compounding during the year.

Frequently an interest rate is described as an annual rate, even though the interest period may be something other than one year. A bank may pay 1\% interest on the amount in a savings account every three months. The nominal interest rate in this situation is 6\% \( (4 \times 1\% = 6\%) \). But if you
deposited $1000 in such an account, would you have 106%(1000) = $1060 in the account at the end of one year? The answer is no, you would have more. The amount in the account would increase as follows:

<table>
<thead>
<tr>
<th>Amount in Account</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>At beginning of year</td>
<td>$1000.00</td>
</tr>
<tr>
<td>End of 3 months:</td>
<td>$1015.00</td>
</tr>
<tr>
<td>End of 6 months:</td>
<td>$1030.23</td>
</tr>
<tr>
<td>End of 9 months:</td>
<td>$1045.68</td>
</tr>
<tr>
<td>End of one year:</td>
<td>$1061.37</td>
</tr>
</tbody>
</table>

The actual interest rate on the $1000 would be the interest, $61.37, divided by the original $1000, or 6.137%. We call this the effective interest rate. The formula for calculating the effective rate is:

\[
\text{Effective interest rate} = (1 + i)^m - 1, \quad \text{where} \quad i = \text{Interest rate per interest period}; \quad m = \text{Number of compoundings per year}.
\]

**EXAMPLE 12**

A bank charges 1½% per month on the unpaid balance for purchases made on its credit card. What nominal interest rate is it charging? What effective interest rate?

**Solution**

The nominal interest rate is simply the annual interest ignoring compounding, or 12(1½%) = 18%. Effective interest rate = \((1 + 0.015)^{12} - 1 = 0.1956 = 19.56\%\)

**SOLVING ECONOMIC ANALYSIS PROBLEMS**

The techniques presented so far illustrate how to convert single amounts of money, and uniform or gradient series of money, into some equivalent sum at another point in time. These compound interest computations are an essential part of economic analysis problems.

The typical situation is that we have a number of alternatives and the question is, which alternative should be selected? The customary method of solution is to resolve each of the alternatives into some common form and then choose the best alternative (taking both the monetary and intangible factors into account).

**Criteria**

Economic analysis problems inevitably fall into one of these categories:

1. **Fixed Input**

   The amount of money or other input resources is fixed.

   **Example:** A project engineer has a budget of $450,000 to overhaul a plant.

2. **Fixed Output**

   There is a fixed task, or other output to be accomplished.
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Example: A consulting engineer must purchase a new computer.

3. Neither Input nor Output Fixed

This is the general situation where neither the amount of money or other inputs, nor the amount of benefits or other outputs are fixed.

Example: A mechanical engineering firm has more work available than it can handle. It is considering paying the staff for working evenings to increase the amount of design work it can perform.

There are five major methods of comparing alternatives: present worth; annual worth; future worth; rate of return; and benefit/cost ratio. These are presented in the following sections.

PRESENT WORTH

In present worth analysis, the approach is to resolve all the money consequences of an alternative into an equivalent present sum. For the three categories given above, the criteria are:

<table>
<thead>
<tr>
<th>Category</th>
<th>Present Worth Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Input</td>
<td>Maximize the Present Worth of benefits or other outputs.</td>
</tr>
<tr>
<td>Fixed Output</td>
<td>Minimize the Present Worth of costs or other inputs.</td>
</tr>
<tr>
<td>Neither Input nor Output</td>
<td>Maximize [Present Worth of benefits minus Present Worth of costs] or, stated another way: Maximize Net Present Worth.</td>
</tr>
</tbody>
</table>

Application of Present Worth

Present worth analysis is most frequently used to determine the present value of future money receipts and disbursements. We might want to know, for example, the present worth of an income producing property, like an oil well. This should provide an estimate of the price at which the property could be bought or sold.

An important restriction in the use of present worth calculations is that there must be a common analysis period when comparing alternatives. It would be incorrect, for example, to compare the present worth (PW) of cost of Pump A, expected to last 6 years, with the PW of cost of Pump B, expected to last 12 years.

In situations like this, the solution is either to use some other analysis technique (generally the annual worth method) or to restructure the problem so there is a common analysis period. In the
example above, a customary assumption would be that a pump is needed for 12 years and that Pump A will be replaced by an identical Pump A at the end of 6 years. This gives a 12-year common analysis period.

This approach is easy to use when the different lives of the alternatives have a practical least common multiple life. When this is not true (for example, life of J equals 7 years and the life of K equals 11 years), some assumptions must be made to select a suitable common analysis period, or the present worth method should not be used.

EXAMPLE 13
Machine X has an initial cost of $10,000, annual maintenance of $500 per year, and no salvage value at the end of its four-year useful life. Machine Y costs $20,000. The first year there is no maintenance cost. The second year, maintenance is $100, and increases $100 per year in subsequent years. The machine has an anticipated $5000 salvage value at the end of its 12-year useful life. If interest is 8%, which machine should be selected?

Solution
The analysis period is not stated in the problem. Therefore we select the least common multiple of the lives, or 12 years, as the analysis period.

Present Worth of Costs Machine X

\[
= 10,000 + 10,000(P/F, 8\%, 4) + 10,000(P/F, 8\%, 8) + 500(P/A, 8\%, 12) \\
= 10,000 + 10,000(0.7350) + 10,000(0.5403) + 500(7.536) \\
= $26,521
\]

Present Worth of Costs Machine Y

\[
= 20,000 + 100(P/G, 8\%, 12) - 5000(P/F, 8\%, 12) \\
= 20,000 + 100(34.634) - 5000(0.3971) \\
= $21,478
\]

Choose Machine Y with its smaller PW of Costs.

EXAMPLE 14
Two alternatives have the following cash flows:

<table>
<thead>
<tr>
<th>Year</th>
<th>Alternative</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2000</td>
<td>-2800</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>+800</td>
<td>+1100</td>
<td></td>
</tr>
</tbody>
</table>
At a 5% interest rate, which alternative should be selected?

**Solution**

Solving by Present Worth analysis:

Net Present Worth (NPW) = PW of benefits - PW of costs

\[
\text{NPW}_A = 800(P/A, 5\%, 3) - 2000 = 800(2.723) - 2000 = $178.40
\]

\[
\text{NPW}_B = 1100(P/A, 5\%, 3) - 2800 = 1100(2.723) - 2800 = $195.30
\]

To maximize NPW, choose Alternative B.

### Capitalized Cost

In the special situation where the analysis period is infinite \((n = \infty)\), an analysis of the present worth of cost is called **capitalized cost**. There are a few public projects where the analysis period is infinity. Other examples would be permanent endowments and cemetery perpetual care.

When \(n\) equals infinity, a present sum \(P\) will accrue interest of \(Pi\) for every future interest period. For the principal sum \(P\) to continue undiminished (an essential requirement for \(n\) equal to infinity), the end-of-period sum \(A\) that can be disbursed is \(Pi\).

\[
P \rightarrow P + Pi \rightarrow P + Pi \rightarrow P + Pi \rightarrow \cdots \rightarrow A \rightarrow A \rightarrow A \rightarrow \cdots
\]

When \(n = \infty\), the fundamental relationship between \(P\), \(A\), and \(i\) is

\[A = Pi\]

Some form of this equation is used whenever there is a problem with an infinite analysis period.

**EXAMPLE 15**

In his will, a man wishes to establish a perpetual trust to provide for the maintenance of a small local park. If the annual maintenance is $7500 per year and the trust account can earn 5% interest, how much money must be set aside in the trust?

**Solution**

When \(n = \infty\), \(A = Pi\) or \(P = A/i\)
ANNUAL WORTH

The annual worth method is more accurately described as the method of Equivalent Uniform Annual Cost (EUAC) or, where the computation is of benefits, the method of Equivalent Uniform Annual Benefits (EUAB).

For each of the three possible categories of problems, there is an annual worth criterion for economic efficiency.

<table>
<thead>
<tr>
<th>Category</th>
<th>Annual Worth Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Input</td>
<td>Maximize the Equivalent Uniform Annual Benefits. That is, maximize EUAB.</td>
</tr>
<tr>
<td>Fixed Output</td>
<td>Minimize the Equivalent Uniform Annual Cost. That is, minimize EUAC.</td>
</tr>
<tr>
<td>Neither Input nor Output Fixed</td>
<td>Maximize EUAB - EUAC. Or stated another way maximize Annual Worth.</td>
</tr>
</tbody>
</table>

**Application of Annual Worth Analysis**

In the section on present worth, we pointed out that the present worth method requires that there be a common analysis period for all alternatives. This same restriction does not apply in all annual cost calculations, but it is important to understand the circumstances that justify comparing alternatives with different service lives.

Frequently an analysis is to provide for a more or less continuing requirement. One might need to pump water from a well, for example, as a continuing requirement. Regardless of whether the pump has a useful service life of 6 years or 12 years, we would select the one whose annual cost is a minimum. And this would still be the case if the pump useful lives were the more troublesome 7 and 11 years, respectively. Thus, if we can assume a continuing need for an item, an annual cost comparison among alternatives of differing service lives is valid.

The underlying assumption made in these situations is that when the shorter-lived alternative has reached the end of its useful life, it can be replaced with an identical item with identical costs, and so forth. This means the annual worth of the initial alternative is equal to the annual worth for the continuing series of replacements. The underlying “continuing requirement” assumption is often present, so the annual worth comparison of unequal-lived alternatives is an appropriate method of analysis.

If, on the other hand, there is a specific requirement in some situation to pump water for 10 years, then each pump must be evaluated to see what costs will be incurred during the analysis period and what salvage value, if any, may be recovered at the end of the analysis period. The annual worth comparison needs to consider the actual circumstances of the situation.

**EXAMPLE 16**

Consider the following alternatives:

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>First cost</td>
<td>$5,000</td>
<td>$10,000</td>
</tr>
<tr>
<td>Annual maintenance</td>
<td>500</td>
<td>200</td>
</tr>
<tr>
<td>End-of-useful-life salvage</td>
<td>600</td>
<td>1,000</td>
</tr>
</tbody>
</table>
Based on an 8% interest rate, which alternative should be selected?

**Solution**

Assuming both alternatives perform the same task and there is a continuing requirement, the goal is to minimize EUAC.

*Alternative A:*

\[
\text{EUAC} = 5,000(A/P, 8\%, 5) + 500 - 600(A/F, 8\%, 5) \\
= 5,000(0.2505) + 500 - 600(0.1705) = $1,650
\]

*Alternative B:*

\[
\text{EUAC} = 10,000(A/P, 8\%, 15) + 200 - 1000(A/F, 8\%, 15) \\
= 10,000(0.1168) + 200 - 1,000(0.0368) = $1,331
\]

To minimize EUAC, select Alternative B.

**FUTURE WORTH**

In present worth analysis, the comparison is made in terms of the equivalent present costs and benefits. But the analysis need not be made at the present time. It could be made at any point in time: past, present, or future. Although the numerical calculations may look different, the decision is unaffected by the point in time selected. Of course, there are situations where we do want to know what the future situation will be if we take some particular course of action now. When an analysis is made based on some future point in time, it is called future worth analysis.

**Category**

<table>
<thead>
<tr>
<th>Category</th>
<th>Future Worth Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Input</td>
<td>Maximize the Future Worth of benefits or other outputs.</td>
</tr>
<tr>
<td>Fixed Output</td>
<td>Minimize the Future Worth of costs or other inputs.</td>
</tr>
<tr>
<td>Neither Input nor Output Fixed</td>
<td>Maximize [Future Worth of benefits minus Future Worth of costs] or, stated another way: Maximize Net Future Worth.</td>
</tr>
</tbody>
</table>

**EXAMPLE 17**

Two alternatives have the following cash flows:

<table>
<thead>
<tr>
<th>Year</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$2000</td>
<td>-$2800</td>
</tr>
<tr>
<td>1</td>
<td>+800</td>
<td>+1100</td>
</tr>
<tr>
<td>2</td>
<td>+800</td>
<td>+1100</td>
</tr>
<tr>
<td>3</td>
<td>+800</td>
<td>+1100</td>
</tr>
</tbody>
</table>

At a 5% interest rate, which alternative should be selected?

**Solution**

In Example 14, this problem was solved by Present Worth analysis at Year 0. Here it will be solved by Future Worth analysis at the end of Year 3.

Net Future Worth (NFW) = FW of benefits - FW of cost

\[
NFW_{A} = 800(F/A, 5\%, 3) - 2000(F/P, 5\%, 3)
\]
To maximize NPV, choose Alternative B.

**RATE OF RETURN**

A typical situation is a cash flow representing the costs and benefits. The rate of return may be defined as the interest rate where

\[
PW_{\text{benefits}} = PW_{\text{costs}} \quad \text{or} \quad PW_{\text{benefits}} - PW_{\text{costs}} = 0
\]

or

\[EUAB = EUAC \quad \text{or} \quad EUAB - EUAC = 0\]

**EXAMPLE 18**

Compute the rate of return for the investment represented by the following cash flow:

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$595</td>
</tr>
<tr>
<td>1</td>
<td>+250</td>
</tr>
<tr>
<td>2</td>
<td>+200</td>
</tr>
<tr>
<td>3</td>
<td>+150</td>
</tr>
<tr>
<td>4</td>
<td>+100</td>
</tr>
<tr>
<td>5</td>
<td>+50</td>
</tr>
</tbody>
</table>

**Solution**

This declining arithmetic gradient series may be separated into two cash flows for which compound interest factors are available:

Note that the gradient series factors are based on an *increasing* gradient. Here, subtracting an increasing arithmetic gradient, as indicated by the diagram, solves the declining cash flow.

\[
PW \text{ of benefits} - PW \text{ of costs} = 0
\]

\[\left(250 \left(\frac{P}{A}, i, 5\right) - 50 \left(\frac{P}{G}, i, 5\right)\right) - 595 = 0\]
Try \( i = 10 \% \):
\[
[250(3.791) - 50(6.862)] - 595 = -9.65
\]

Try \( i = 12 \% \):
\[
[250(3.605) - 50(6.397)] - 595 = +13.60
\]

The rate of return is between 10\% and 12\%. It may be computed more accurately by linear interpolation:

\[
\text{Rate of return} = 10\% + 2\% \left( \frac{9.65 - 0}{13.60 - (-9.65)} \right) = 10.83\%
\]

**Application of Rate of Return with Two Alternatives**

To properly compare two alternatives using rate of return, compute the incremental rate of return on the cash flow representing the difference between the two alternatives. Since we want to look at increments of investment, the cash flow for the difference between the alternatives is computed by taking the higher initial-cost alternative minus the lower initial-cost alternative. If the incremental rate of return is greater than or equal to the predetermined minimum attractive rate of return (MARR), choose the higher-cost alternative; otherwise, choose the lower-cost alternative.

**EXAMPLE 19**

Two alternatives have the following cash flows:

<table>
<thead>
<tr>
<th>Year</th>
<th>Alternative A</th>
<th>Alternative B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$2000</td>
<td>-$2800</td>
</tr>
<tr>
<td>1</td>
<td>+800</td>
<td>+1100</td>
</tr>
<tr>
<td>2</td>
<td>+800</td>
<td>+1100</td>
</tr>
<tr>
<td>3</td>
<td>+800</td>
<td>+1100</td>
</tr>
</tbody>
</table>

If 5\% is considered the minimum attractive rate of return (MARR), which alternative should be selected?

**Solution**

These two alternatives were previously examined in Examples 14 and 17 by present worth and future worth analysis. This time, the alternatives will be resolved using rate of return analysis.

Note that the problem statement specifies a 5\% minimum attractive rate of return (MARR), while Examples 14 and 16 referred to a 5\% interest rate. These are really two different ways of saying the same thing: the minimum acceptable time value of money is 5\%.

First, tabulate the cash flow that represents the increment of investment between the alternatives. Taking the higher initial-cost alternative minus the lower initial-cost alternative does this:

<table>
<thead>
<tr>
<th>Year</th>
<th>Alternative A</th>
<th>Alternative B</th>
<th>Difference between Alternatives</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>( B - A )</td>
</tr>
</tbody>
</table>
Then compute the rate of return on the increment of investment represented by the difference between the alternatives:

\[
PW \text{ of benefits} = PW \text{ of costs} \\
300(P/A, i, 3) = 800 \\
(P/A, i, 3) = 800/300 \\
i \approx 6.1\%
\]

Since the incremental rate of return exceeds the 5% MARR, the increment of investment is desirable. Choose the higher-cost Alternative B.

Before leaving this example problem, one should note something that relates to the rates of return on Alternative A and on Alternative B. These rates of return, if computed, are:

<table>
<thead>
<tr>
<th>Rate of Return</th>
<th>Alternative A</th>
<th>9.7%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alternative B</td>
<td>8.7%</td>
</tr>
</tbody>
</table>

The correct answer to this problem has been shown to be Alternative B, and this is true even though Alternative A has a higher rate of return. The higher-cost alternative may be thought of as the lower cost alternative, plus the increment of investment between them. Looked at this way, the higher-cost Alternative B is equal to the desirable lower-cost Alternative A plus the desirable differences between the alternatives.

The important conclusion is that computing the rate of return for each alternative does not provide the basis for choosing between alternatives. Instead, incremental analysis is required.

**EXAMPLE 20**
Consider the following:

<table>
<thead>
<tr>
<th>Year</th>
<th>Alternative A</th>
<th>Alternative B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$200.0</td>
<td>-$131.0</td>
</tr>
<tr>
<td>1</td>
<td>+77.6</td>
<td>+48.1</td>
</tr>
<tr>
<td>2</td>
<td>+77.6</td>
<td>+48.1</td>
</tr>
<tr>
<td>3</td>
<td>+77.6</td>
<td>+48.1</td>
</tr>
</tbody>
</table>

If the minimum attractive rate of return (MARR) is 10%, which alternative should be selected?

**Solution**
To examine the increment of investment between the alternatives, we will examine the higher initial cost alternative minus the lower initial-cost alternative, or \( A - B \).

<table>
<thead>
<tr>
<th>Year</th>
<th>Alternative A</th>
<th>Alternative B</th>
<th>Increment A - B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$200.0</td>
<td>-$131.0</td>
<td>-$69.0</td>
</tr>
<tr>
<td>1</td>
<td>+77.6</td>
<td>+48.1</td>
<td>+29.5</td>
</tr>
<tr>
<td>2</td>
<td>+77.6</td>
<td>+48.1</td>
<td>+29.5</td>
</tr>
</tbody>
</table>
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<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>+77.6</td>
<td>+48.1</td>
</tr>
</tbody>
</table>

Solve for the incremental rate of return:

\[
PW \text{ of benefits} = PW \text{ of costs}
\]
\[
29.5(P/A, i, 3) = 69.0
\]
\[
(P/A, i, 3) = 69.0/29.5 = 2.339
\]

From the Compound Interest Tables, 12% < ROR\_{A-B} < 15%, Therefore we select the higher initial-cost Alternative \(A\).

**Application of Rate of Return with Three or More Alternatives**

When there are three or more mutually exclusive alternatives, one must proceed following the same general logic presented for two alternatives. The components of incremental analysis are:

1. Compute the rate of return for each alternative. Reject any alternative where the rate of return is less than the given MARR.
2. Rank the remaining alternatives in their order of increasing initial cost.
3. Examine the increment of investment between the two lowest-cost alternatives as described for the two-alternative problem. Select the best of the two alternatives and reject the other one.
4. Take the preferred alternative from Step 3. Consider the next higher initial-cost alternative and proceed with another two-alternative comparison. (An alternative is to use the two highest cost alternatives first.)
5. Continue until all alternatives have been examined and the best of the multiple alternatives has been identified.

**EXAMPLE 21**

Consider the following:

<table>
<thead>
<tr>
<th>Year</th>
<th>Alternative A</th>
<th>Alternative B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$200.0</td>
<td>-$131.0</td>
</tr>
<tr>
<td>1</td>
<td>+77.6</td>
<td>+48.1</td>
</tr>
<tr>
<td>2</td>
<td>+77.6</td>
<td>+48.1</td>
</tr>
<tr>
<td>3</td>
<td>+77.6</td>
<td>+48.1</td>
</tr>
</tbody>
</table>

If the minimum attractive rate of return (MARR) is 10%, which alternative, if any, should be selected?

**Solution**

One should carefully note that this is a three-alternative problem where the alternatives are \(A\), \(B\), and “Do Nothing”. (Neither \(A\) nor \(B\) must be chosen.)
In this solution we will skip Step 1 because we have previously solved for the ROR_A and ROR_B. Reorganize the problem by placing the alternatives in order of increasing initial cost:

<table>
<thead>
<tr>
<th>Alternative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

Examine the “B - Do Nothing” increment of investment:

| Year | B - Nothing | Do |
| 0   | -$131.0 - 0 = -$131.0 |
| 1   | +48.1 - 0 = +48.1 |
| 2   | +48.1 - 0 = +48.1 |
| 3   | +48.1 - 0 = +48.1 |

Solve for the incremental rate of return:

\[
\text{PW of benefits} = \text{PW of costs} \\
131.0 = 48.1(P/A, i, 3) \\
(P/A, i, 3) = 131.0/48.1 = 2.723
\]

From Compound Interest Tables, the incremental rate of return = 5%. Since the incremental rate of return is less than 10%, the B - Do Nothing increment is not desirable. Reject Alternative B.

Next, consider the increment of investment between the two remaining alternatives:

| Year | A - Nothing | Do |
| 0   | -$200.0 - 0 = -$200.0 |
| 1   | +77.6 - 0 = +77.6 |
| 2   | +77.6 - 0 = +77.6 |
| 3   | +77.6 - 0 = +77.6 |

Solve for the incremental rate of return:

\[
\text{PW of benefits} = \text{PW of costs} \\
200.0 = 77.6(P/A, i, 3) \\
(P/A, i, 3) = 200/77.6 = 2.577
\]

From Compound Interest Tables, the incremental rate of return is 8%. Since the rate of return on the A - Do Nothing increment of investment is less than the desired 10%, reject the increment by rejecting Alternative A. We select the remaining alternative: Do nothing!

If you have not already done so, you should go back to Example 20 and see how the slightly changed wording of the problem radically altered it. Example 20 required the choice between two undesirable alternatives. Example 21 adds the Do-nothing alternative that is superior to A or B.

**EXAMPLE 22**

Consider four mutually exclusive alternatives:

| Alternative |
Each alternative has a five-year useful life and no salvage value. If the minimum attractive rate of return (MARR) is 6%, which alternative should be selected?

**Solution**

*Mutually exclusive* is where selecting one alternative precludes selecting any of the other alternatives. This is the typical “textbook” situation. The solution will follow the several steps in incremental analysis.

1. The rate of return is computed for the four alternatives.

<table>
<thead>
<tr>
<th>Alternative</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computed rate of return</td>
<td>8.3%</td>
<td>11.9%</td>
<td>5%</td>
<td>8%</td>
</tr>
</tbody>
</table>

   Since ROR_C < MARR, it may be eliminated from further consideration.

2. Rank the remaining alternatives in order of increasing initial cost and examine the increment between the two lowest cost alternatives.

<table>
<thead>
<tr>
<th>Alternative</th>
<th>B</th>
<th>A</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Cost</td>
<td>$100.0</td>
<td>$400.0</td>
<td>$500.0</td>
</tr>
<tr>
<td>Uniform Annual Benefit</td>
<td>27.7</td>
<td>100.9</td>
<td>125.2</td>
</tr>
</tbody>
</table>

   $A - B$

   | \( \Delta \) Initial Cost | $300.0 |
   | \( \Delta \) Uniform Annual Benefit | 73.2 |
   | \( \Delta \) ROR_{A-B} | 7% |

   Since the incremental ROR exceeds the 6% MARR, Alternative A is the better alternative.

3. Take the preferred alternative from the previous step and consider the next higher-cost alternative. Do another two-alternative comparison.

   $D - A$

   | \( \Delta \) Initial Cost | $100.0 |
   | \( \Delta \) Uniform Annual Benefit | 24.3 |
   | \( \Delta \) ROR_{D-A} | 6.9% |

   The incremental ROR exceeds the 6% MARR, Alternative D is preferred over Alternative A.

Conclusion: Select Alternative D. Note that once again the alternative with the highest rate of return (Alt. B) is not the proper choice.

**BENEFIT/COST RATIO**
Generally in public works and governmental economic analyses, the dominant method of analysis is called benefit/cost ratio. It is simply the ratio of benefits divided by costs, taking into account the time value of money.

\[
B / C = \frac{\text{PW of benefits}}{\text{PW of costs}} = \frac{\text{Equivalent Uniform Annual Benefits}}{\text{Equivalent Uniform Annual Costs}}
\]

For a given interest rate, a B/C ratio \( \geq 1 \) reflects an acceptable project. The method of analysis using B/C ratio is parallel to that of rate of return analysis. The same kind of incremental analysis is required.

### Application of B/C Ratio for Two Alternatives

Compute the incremental B/C ratio for the cash flow representing the increment of investment between the higher initial-cost alternative and the lower initial-cost alternative. If this incremental B/C ratio is \( \geq 1 \), choose the higher-cost alternative; otherwise, choose the lower-cost alternative.

### Application of B/C Ratio for Three or More Alternatives

Follow the procedure used for rate of return, except that the test is whether or not the incremental B/C ratio is \( \geq 1 \).

**EXAMPLE 23**

Solve Example 22 using benefit/cost analysis. Consider four mutually exclusive alternatives:

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Initial Cost</th>
<th>Uniform Annual Benefit</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$400.0</td>
<td>100.9</td>
</tr>
<tr>
<td>B</td>
<td>$100.0</td>
<td>27.7</td>
</tr>
<tr>
<td>C</td>
<td>$200.0</td>
<td>46.2</td>
</tr>
<tr>
<td>D</td>
<td>$500.0</td>
<td>125.2</td>
</tr>
</tbody>
</table>

Each alternative has a five-year useful life and no salvage value. Based on a 6% interest rate, which alternative should be selected?

**Solution**

1. B/C ratio computed for the alternatives:

   Alt. A
   \[
   B / C = \frac{\text{PW of benefits}}{\text{PW of costs}} = \frac{100.9(\text{P/A, 6\%, 5})}{400} = 1.06
   \]

   Alt. B
   \[
   B / C = \frac{\text{PW of benefits}}{\text{PW of costs}} = \frac{27.7(\text{P/A, 6\%, 5})}{100} = 1.17
   \]

   Alt. C
   \[
   B / C = \frac{\text{PW of benefits}}{\text{PW of costs}} = \frac{46.2(\text{P/A, 6\%, 5})}{200} = 0.97
   \]
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\[ \text{Alt. } D \quad \frac{B}{C} = \frac{\text{PW of benefits}}{\text{PW of cost}} = \frac{125.2(A, 6\%, 5)}{500} = 1.05 \]

Alternative C with a B/C ratio less than 1 is eliminated.

2. Rank the remaining alternatives in order of increasing initial cost and examine the increment of investment between the two lowest cost alternatives.

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Initial Cost</th>
<th>Uniform Annual Benefit</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$100.0</td>
<td>27.7</td>
</tr>
<tr>
<td>A</td>
<td>$400.0</td>
<td>100.9</td>
</tr>
<tr>
<td>D</td>
<td>$500.0</td>
<td>125.2</td>
</tr>
</tbody>
</table>

\[ \Delta \text{Initial Cost} = \text{A - B} = 300.0 \]
\[ \Delta \text{Uniform Annual Benefit} = 73.2 \]

Incremental B/C ratio \[= \frac{73.2(A, 6\%, 5)}{300} = 1.03 \]

The incremental B/C ratio exceeds 1.0 therefore Alternative A is preferred over B.

3. Take the preferred alternative from the previous step and consider the next higher-cost alternative. Do another two-alternative comparison.

\[ \Delta \text{Initial Cost} = \text{D - A} = 100.0 \]
\[ \Delta \text{Uniform Annual Benefit} = 24.3 \]

Incremental B/C ratio \[= \frac{24.3(A, 6\%, 5)}{100} = 1.02 \]

The incremental B/C ratio exceeds 1.0 therefore Alternative D is preferred over A.

Conclusion: Select Alternative D.

BREAKEVEN ANALYSIS

In business, “breakeven” is defined as the point where income just covers the associated costs. In engineering economics, the breakeven point is more precisely defined as the point where two alternatives are equivalent.

EXAMPLE 24
A city is considering a new $50,000 snowplow. The new machine will operate at a savings of $600 per day, compared to the equipment presently being used. Assume the minimum attractive rate of return is 12% and the machine’s life is 10 years with zero resale value at that time. How many days per year must the machine be used to justify the purchase of the snowplow?
Solution

This breakeven problem may be readily solved by annual cost computations. We will set the equivalent uniform annual cost of the snowplow equal to its equivalent uniform annual benefit, and solve for the required annual utilization.

Let $X =$ breakeven point = days of operation per year.

$$EUAC = EUAB$$

$$50,000(A/P, 12\%, 10) = 600 X$$

$$X = \frac{50,000(0.1770)}{600} = 14.7 \text{ days/year}$$

DEPRECIATION

Depreciation of capital equipment is an important component of after-tax economic analysis. For this reason, one must understand the fundamentals of depreciation accounting.

*Depreciation* is defined in its accounting sense, as the systematic allocation of the cost of a capital asset over its useful life. *Book value* is defined as the original cost of an asset, minus the accumulated depreciation of the asset.

In computing a schedule of depreciation charges, three items are considered:

1. Cost of the property, $P$;
2. Depreciable life in years, $n$;
3. Salvage value of the property at the end of its depreciable life, $S$.

**Straight Line Depreciation**

Depreciation charge in any year = $\frac{P - S}{n}$

**Sum-Of-Years-Digits Depreciation**

Depreciation charge in any year = $\frac{\text{Remaining Depreciable Life at Beginning of Year}}{\text{Sum of Years Digits for Total Useful Life}} \times (P - S)$

where $\text{Sum Of Years Digits} = 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$

**Double Declining Balance Depreciation**
Depreciation charge in any year = \( \frac{2}{n} (P - \text{Depreciation charges to date}) \)

### Modified Accelerated Cost Recovery System Depreciation

Modified accelerated cost recovery system (MACRS) depreciation is based on a property class life that is typically less than the actual useful life of the property and on zero salvage value. The varying depreciation percentage to use must be read from a table (based on declining balance with conversion to straight line). The proper MACRS property class must first be determined in order to calculate the depreciation charge in any given year. The following tables are used with MACRS depreciation. Table 1 categorizes assets by recovery period. Table 2 provides the percentage depreciation allowed each year.

Depreciation charge in any year \( n = P (\text{percentage depreciation allowance}) \)

#### Table 1 Recovery Periods for MACRS

<table>
<thead>
<tr>
<th>Property Class</th>
<th>Personal Property (all except real estate)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-Year</td>
<td>Special handling devices for food and beverage manufacture</td>
</tr>
<tr>
<td></td>
<td>Special tools for the manufacture of finished plastic products, fabricated metal products, and motor vehicles</td>
</tr>
<tr>
<td></td>
<td>Property with ADR &lt; 4 years</td>
</tr>
<tr>
<td>5-Year</td>
<td>Automobiles, buses, trucks, non-commercial aircraft</td>
</tr>
<tr>
<td></td>
<td>Computers</td>
</tr>
<tr>
<td></td>
<td>Construction equipment, petroleum drilling equipment and R&amp;D equipment</td>
</tr>
<tr>
<td></td>
<td>Property with 4 years &lt; ADR &lt; 10 years</td>
</tr>
<tr>
<td>7-Year</td>
<td>And items not assigned to another class</td>
</tr>
<tr>
<td></td>
<td>Office furniture, fixture, and equipment</td>
</tr>
<tr>
<td></td>
<td>Most manufacturing equipment and mining equipment</td>
</tr>
<tr>
<td></td>
<td>Property with 10 years ≤ ADR &lt; 16 years</td>
</tr>
<tr>
<td>10-Year</td>
<td>Marine vessels and water transportation equipment</td>
</tr>
<tr>
<td></td>
<td>Petroleum refining equipment</td>
</tr>
<tr>
<td></td>
<td>Single-purpose agricultural structures, trees and vines that bear nuts or fruits</td>
</tr>
<tr>
<td></td>
<td>Property with 16 years ≤ ADR &lt; 20 years</td>
</tr>
<tr>
<td>15-Year</td>
<td>Steam and electric generation and distribution systems</td>
</tr>
<tr>
<td></td>
<td>Telephone distribution facilities</td>
</tr>
<tr>
<td></td>
<td>Municipal wastewater (sewage) treatment facilities</td>
</tr>
<tr>
<td></td>
<td>Property with 20 years ≤ ADR &lt; 25 years</td>
</tr>
<tr>
<td>20-Year</td>
<td>Municipal sewers</td>
</tr>
<tr>
<td></td>
<td>Property with ADR ≥ 25 years</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Property Class</th>
<th>Real Property (real estate)</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.5-Year</td>
<td>Residential rental property (Does not include hotels and motels)</td>
</tr>
<tr>
<td>39-Year</td>
<td>Non-residential real property</td>
</tr>
</tbody>
</table>
## Table 2 MACRS Percentages

<table>
<thead>
<tr>
<th>Year</th>
<th>3-year</th>
<th>5-year</th>
<th>7-year</th>
<th>10-year</th>
<th>15-year</th>
<th>20-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>33.33</td>
<td>20.00</td>
<td>14.29</td>
<td>10.00</td>
<td>5.00</td>
<td>3.750</td>
</tr>
<tr>
<td>2</td>
<td>44.45</td>
<td>32.00</td>
<td>24.49</td>
<td>18.00</td>
<td>9.50</td>
<td>7.219</td>
</tr>
<tr>
<td>3</td>
<td>14.81</td>
<td>19.20</td>
<td>17.49</td>
<td>14.40</td>
<td>8.55</td>
<td>6.677</td>
</tr>
<tr>
<td>4</td>
<td>7.41</td>
<td>11.52</td>
<td>12.49</td>
<td>11.52</td>
<td>7.70</td>
<td>6.177</td>
</tr>
<tr>
<td>5</td>
<td>11.52</td>
<td>8.93</td>
<td>9.22</td>
<td>6.93</td>
<td>5.713</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5.76</td>
<td>8.92</td>
<td>7.37</td>
<td>6.23</td>
<td>5.285</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8.93</td>
<td>6.55</td>
<td>5.90</td>
<td>4.888</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4.46</td>
<td>6.55</td>
<td>5.90</td>
<td>4.522</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>6.56</td>
<td>5.91</td>
<td>4.462</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>6.55</td>
<td>5.90</td>
<td>4.461</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>3.28</td>
<td>5.91</td>
<td>4.462</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>5.90</td>
<td>4.461</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>5.91</td>
<td>4.462</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>5.90</td>
<td>4.461</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>5.91</td>
<td>4.462</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>2.95</td>
<td>4.461</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td>4.462</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td>4.461</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td></td>
<td>4.462</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>4.461</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td></td>
<td>2.231</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**EXAMPLE 25**

A piece of construction machinery costs $5000 and has an anticipated $1000 salvage value at the end of its five-year depreciable life. Compute the depreciation schedule for the machinery by:

(a) Straight-line depreciation;
(b) Sum-of-years-digits depreciation;
(c) Double declining balance depreciation.
(d) MACRS

**Solution**

Straight-line depreciation = \( \frac{P - S}{n} = \frac{5000 - 1000}{5} = $800 \)

Sum-of-years-digits depreciation:

\[
\text{Sum-of-years-digits} = \frac{n}{2}(n + 1) = \frac{5}{2}(6) = 15
\]
30 Course Summary

1st-year depreciation = \( \frac{5}{15} (5000 - 1000) = \$1333 \)

2nd-year depreciation = \( \frac{4}{15} (5000 - 1000) = \$1067 \)

3rd-year depreciation = \( \frac{3}{15} (5000 - 1000) = \$800 \)

4th-year depreciation = \( \frac{2}{15} (5000 - 1000) = \$533 \)

5th-year depreciation = \( \frac{1}{15} (5000 - 1000) = \$267 \)

Double declining balance depreciation:

1st-year depreciation = \( \frac{2}{5} (5000 - 0) = \$2000 \)

2nd-year depreciation = \( \frac{2}{5} (5000 - 2000) = 1200 \)

3rd-year depreciation = \( \frac{2}{5} (5000 - 3200) = 720 \)

4th-year depreciation = \( \frac{2}{5} (5000 - 3920) = 432 \)

5th-year depreciation = \( \frac{2}{5} (5000 - 4352) = \frac{280}{5} \) 0

$4000

Since the problem specifies a $1000 salvage value, the total depreciation cannot $4000. The double declining balance depreciation must be stopped in the 4th year when it totals $4000.

MACRS

Since the asset is specified as construction equipment the 5-year recovery period is determined. Note that the depreciation will be over a six-year period. This is because only ½ year depreciation is allowed in the first year. The remaining ½ year is recovered in the last year. If an asset is disposed of mid-life, than only ½ of the disposal year’s depreciation is taken. Also note that the salvage value is not used in the depreciation calculations.

1st-year depreciation = \( .200(5000) = \$1000 \)

2nd-year depreciation = \( .320(5000) = 1600 \)

3rd-year depreciation = \( .192(5000) = 960 \)

4th-year depreciation = \( .1152(5000) = 576 \)

5th-year depreciation = \( .1152(5000) = 576 \)

6th-year depreciation = \( .0576(5000) = 288 \)

$5000

The depreciation schedules computed by the four methods are as follows:

<table>
<thead>
<tr>
<th>Year</th>
<th>StL</th>
<th>SOYD</th>
<th>DDB</th>
<th>MACRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$800</td>
<td>$1333</td>
<td>$2000</td>
<td>$1000</td>
</tr>
<tr>
<td>2</td>
<td>800</td>
<td>1067</td>
<td>1200</td>
<td>1600</td>
</tr>
</tbody>
</table>
INCOME TAXES

Income taxes represent another of the various kinds of disbursements encountered in an economic analysis. The starting point in an after-tax computation is the before-tax cash flow. Generally, the before-tax cash flow contains three types of entries:

1. Disbursements of money to purchase capital assets. These expenditures create no direct tax consequence for they are the exchange of one asset (cash) for another (capital equipment).

2. Periodic receipts and/or disbursements representing operating income and/or expenses. These increase or decrease the year-by-year tax liability of the firm.

3. Receipts of money from the sale of capital assets, usually in the form of a salvage value when the equipment is removed. The tax consequence depends on the relationship between the residual value of the asset and its book value (cost - depreciation taken).

<table>
<thead>
<tr>
<th>Residual Value is</th>
<th>Tax Consequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>less than current book value</td>
<td>loss on sale</td>
</tr>
<tr>
<td>equals current book value</td>
<td>no loss &amp; no recapture</td>
</tr>
<tr>
<td>exceeds current book value</td>
<td>recaptured depreciation</td>
</tr>
<tr>
<td>exceeds initial book value</td>
<td>recaptured depreciation &amp; capital gain</td>
</tr>
</tbody>
</table>

After the before-tax cash flow, the next step is to compute the depreciation schedule for any capital assets. Next, taxable income is the taxable component of the before-tax cash flow minus the depreciation. Then, the income tax is the taxable income times the appropriate tax rate. Finally, the after-tax cash flow is the before-tax cash flow adjusted for income taxes.

To organize these data, it is customary to arrange them in the form of a cash flow table, as follows:

<table>
<thead>
<tr>
<th>Year</th>
<th>Before-tax cash flow</th>
<th>Depreciation</th>
<th>Taxable income</th>
<th>Income taxes</th>
<th>After-tax cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

EXAMPLE 26

A corporation expects to receive $32,000 each year for 15 years from the sale of a product. There will be an initial investment of $150,000. Manufacturing and sales expenses will be $8,067 per year. Assume straight-line depreciation, a 15-year useful life and no salvage value. Use a 46% income tax rate. Determine the projected after-tax rate of return.

Solution
Course Summary

Straight line depreciation = \( \frac{P - S}{n} = \frac{150,000 - 0}{15} = $10,000 \) per year

<table>
<thead>
<tr>
<th>Year</th>
<th>Before-tax cash flow</th>
<th>Depreciation</th>
<th>Taxable income</th>
<th>Income taxes</th>
<th>After-tax cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-150,000</td>
<td></td>
<td>-150,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>+23,933</td>
<td>10,000</td>
<td>13,933</td>
<td>-6409</td>
<td>+17,524</td>
</tr>
<tr>
<td>2</td>
<td>+23,933</td>
<td>10,000</td>
<td>13,933</td>
<td>-6409</td>
<td>+17,524</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>+23,933</td>
<td>10,000</td>
<td>13,933</td>
<td>-6409</td>
<td>+17,524</td>
</tr>
</tbody>
</table>

Take the after-tax cash flow and compute the rate of return at which the PW of benefits equals the PW of costs.

\[
17,524(P/A, i, 15) = 150,000 \\
(P/A, i, 15) = 150,000/17,524 \\
= 8.559
\]

From Compound Interest Tables, \( i = 8\% \).

INFLATION

Inflation is a decrease in the buying power of the dollar (or peso, yen, etc.). More generally it is thought of as the increase in the level of prices. Inflation rates are usually estimated as having a constant rate of change. This type of cash flow is a geometric gradient.

Future cash flows are generally estimated in constant-value terms. The assumption is made that the inflation rates for various cash flows will match the economy’s inflation rate. This allows for the use of a real interest rate.

Another approach is the market interest rate. The market rate includes both the time value of money and an estimate of current and predicted inflation.

For exact calculation of real and market interest rates the following formula is used.

\[
(1 + \text{Market rate}) = (1 + i)(1 + f) \\
\text{where } i = \text{the real interest rate} \\
f = \text{the inflation rate}
\]

EXAMPLE 27

Determine the market interest rate if the inflation rate is estimated to be 2.5% and the time value of money (the real interest rate) is 6%.

Solution

\[
(1 + \text{Market rate}) = (1 + .06)(1 + .025) \\
(1 + \text{Market rate}) = 1.0865 \\
\text{Market rate} = 0.0865 = 8.65\%
\]
Uncertainty and Probability

It is unrealistic to consider future cash flows to be known exactly. Cash flows, useful life, salvage value, etc. can all be expressed using probability distributions. Then measures of expected or average return and of risk can be used in the decision making process.

Probabilities can be derived from the following:
1. Historical trends or data.
2. Mathematical models (Probability distributions).
3. Subjective estimates

Probabilities must satisfy the following:
1. $0 < P < 1.$
2. The sum of all probabilities must $= 1.$

Expected values for cash flows, useful life, salvage value, etc., can be found by using a weighted average calculation.

**EXAMPLE 28**
The following table summarizes estimated annual benefits, annual costs, and end-of-life value for an asset under consideration. Determine the expected value for each of the three. If the asset has a first cost of $25,950 and an anticipated life of eight years determine the rate of return for the asset.

<table>
<thead>
<tr>
<th>State and Associated Probability</th>
<th>p = .20</th>
<th>p = .40</th>
<th>p = .25</th>
<th>p = .15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual Benefits</td>
<td>3,000</td>
<td>6,500</td>
<td>8,000</td>
<td>11,500</td>
</tr>
<tr>
<td>Annual Costs</td>
<td>1,500</td>
<td>2,800</td>
<td>5,000</td>
<td>5,750</td>
</tr>
<tr>
<td>End-of-Life Value</td>
<td>5,000</td>
<td>7,500</td>
<td>9,000</td>
<td>10,000</td>
</tr>
</tbody>
</table>

**Solution**

\[
E(\text{Annual Benefits}) = .20(3,000) + .40(6,500) + .25(8,000) + .15(11,500) \\
= $6,925.00
\]

\[
E(\text{Annual Costs}) = .20(1,500) + .40(2,800) + .25(5,000) + .15(5,750) \\
= $3,532.50
\]

\[
E(\text{End-of-Life Value}) = .20(5,000) + .40(7,500) + .25(9,000) + .15(10,000) \\
= $7,750.00
\]

To determine ROR

\[
\text{PW of benefits} - \text{PW of costs} = 0
\]
Course Summary

\[
[6,925(P/A, i\%, 8) + 7,750(P/F, i\%, 8)] - [25,950 + 3,532.5(P/A, i\%, 8)] = 0
\]

Try 6%
\[
[6,925(6.210) + 7,750(.6274)] - [25,950 + 3,532.5(6.210)] = -20.22
\]

Try 5%
\[
[6,925(6.463) + 7,750(.6768)] - [25,950 + 3,532.5(6.463)] = 1,220.93
\]

By interpolation the ROR = 5.98%