69. Assume that $B$ has $m$ left parentheses and $m$ right parentheses and that $C$ has $n$ left parentheses and $n$ right parentheses. How many left parentheses appear in $(B \land C)$? How many right parentheses appear in $(B \land C)$?

70. Following the model given in exercise 69, argue that $(B \lor C)$, $(B \rightarrow C)$, $(B \leftrightarrow C)$ each have the same number of left and right parentheses. Conclude from exercises 67–70 that any sentence has the same number of left and right parentheses.

### 1.2 Truth and Sentential Logic

Mathematicians seek to discover and to understand mathematical truth. The five logical connectives of sentential logic play an important role in determining whether a mathematical statement is true or false. Specifically, the truth value of a compound sentence is determined by the interaction of the truth value of its component sentences and the logical connectives linking these components. In this section, we learn a truth table algorithm for computing all possible truth values of any sentence from sentential logic.

In mathematics we generally assume that every sentence has one of two truth values: true or false. As we discuss in later chapters, the reality of mathematics is far less clear; some sentences are true, some are false, some are neither, while some are unknown. Many questions can be considered in one of the various interesting and reasonable multi-valued logics. For example, philosophers and physicists have successfully utilized multi-valued logics with truth values “true,” “false,” and “unknown” to model and analyze diverse real-world questions. In this book, we keep our study immediately relevant to the most common needs in mathematics by assuming a two-valued logic with truth values “true” denoted by $T$, and “false” denoted by $F$.

In a given setting, one of these two truth values is assigned to each sentence symbol ($A, B, \ldots, Z$), while sentence variables ($a, b, \ldots, z$) are free to assume either truth value. We use truth tables to determine the truth value of sentences built up from sentence symbols, sentence variables, and logical connectives.

We begin by stating the distinct truth table for each logical connective. In defining these basic truth tables, an intuitive understanding of connectives in our natural language drives the interpretation of connectives in the formal language of sentential logic, and so we appeal to our intuition in motivating our formal definitions.

First, consider negation, the “not” connective denoted by $\neg$. Negation switches truth values. For example, if “The number $n$ is prime” is true, then “The number $n$ is not prime” is false; that is, if $P$ is true, then $\neg P$ is false. Similarly, if “The number $n$ is prime” is false, then “The number $n$ is not prime” is true; that is, if $P$ is false, then $\neg P$ is true. We express this analysis both as a phrase to aid memorization and as a truth table.

<table>
<thead>
<tr>
<th>$\neg$ swaps truth values</th>
<th>$P$</th>
<th>$\neg P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
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<tr>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
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</table>
This basic truth table uses the sentence variable $p$, since $p$ (as a variable) is free to assume either truth value $T$ or $F$, enabling a complete analysis of the negation connective. In addition, the truth table has only two rows, since $p$ is the only sentence variable in the sentence $\sim p$.

With this definition in hand, we no longer need to rely on intuition when interpreting the negation connective in a sentence. Instead, the truth table for negation has mathematically formalized the interpretation of negation when computing the truth of sentences. We refer to this truth table when a negation appears in a sentence, an approach which is particularly helpful when working with elaborate compound sentences. By developing similar truth tables for the other logical connectives and capturing our natural intuitions about these connectives, we establish the complete tools for developing an algebra of truth for sentential logic.

Turning to the other connectives, consider conjunction, the “and” connective denoted $\land$. We interpret $p \land q$ as true exactly when both $p$ and $q$ are true. If $p$ is false or if $q$ is false or if both $p$ and $q$ are false, then $p \land q$ is false. As above, we gather this analysis (and the results of a similar analysis for the other connectives) into a collection of phrases and truth tables.

<table>
<thead>
<tr>
<th>$\land$ is $T$ if</th>
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<tbody>
<tr>
<td>both $T$ and</td>
</tr>
<tr>
<td>$F$ otherwise</td>
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</table>

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
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<tr>
<td>both $F$ and</td>
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<td>$T$ otherwise</td>
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<th>$p$</th>
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<th>$\rightarrow$ is $F$ if</th>
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<tr>
<td>$T \rightarrow F$ and</td>
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<tr>
<td>$T$ otherwise</td>
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<table>
<thead>
<tr>
<th>$\leftrightarrow$ is $T$ if</th>
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<tr>
<td>the same and</td>
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<tr>
<td>$F$ otherwise</td>
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<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \leftrightarrow q$</th>
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Since each sentence in the above chart has two sentence variables, there are four rows in each truth table. In particular, each sentence variable can be either true or false, resulting in the four possible permutations of truth values: $TT$, $TF$, $FT$, $FF$. The left columns in each truth table list these four possibilities. We think of a truth table with permutations $TT$, $TF$, $FT$, $FF$ (in this order) as the standard truth table for a sentence with two variables. You should mirror this pattern in your truth table computations to facilitate comparisons among sentences.

The truth tables for disjunction and implication warrant further comment. For the disjunction $p \lor q$, note that there are two standard yet very different usages of the word “or” in our natural language of English. For example, suppose you are eating at your favorite fast food restaurant and the server asks you two questions:

- Would you like french fries or onion rings with your value meal?
- Would you like cream or sugar with your coffee?
In response to the fries–rings question, you can ask for fries or for onion rings, but not both, and you would not be upset that you can only have one; we refer to this use of disjunction as an exclusive-or. In contrast, in response to the cream–sugar question, you can ask for cream or sugar or both, and opting for both is a common choice among coffee lovers; we refer to this use of disjunction as an inclusive-or. In everyday life, context and social norms typically clarify this potential ambiguity in the use of “or.” However, for our formal language, we must avoid such ambiguity and choose just one of these two options as the standard for all disjunctions. Over time, mathematicians and philosophers have adopted the inclusive-or as the standard interpretation of “or,” and so we define \( p \lor q \) as true when \( p \) is true, when \( q \) is true, or when both \( p \) and \( q \) are true.

In standard mathematical practice, the implication \( p \rightarrow q \) is the most important logical connective. Mathematics is essentially a science of implications in which we explicitly identify assumptions and establish the conditional truth of mathematical statements. The first two lines of the truth table for implication match most people’s intuitions: “true implies true” is true and “true implies false” is false. But, why should “false implies true” or “false implies false” be defined as a true statement?

A couple of examples may clarify this choice. First, consider a common “bribe” offered by parents to their children: “If you behave in the store, then we will stop for ice cream.” If the child does not behave in the store, the parents’ statement would be considered true not only if they do not stop for ice cream (the “false implies false” case), but even if, in a moment of benevolent generosity, they do stop for ice cream (the “false implies true” case). In particular, the parent’s statement is false only when the child behaves in the store, but they do not stop for ice cream (the “true implies false” case). Similar situations arise quite frequently in mathematics. For example, consider the assertion “If \( n \geq 3 \), then \( n^2 \geq 4 \).” This statement is true even for \( n = 1 \), when \( n \geq 3 \) is false and \( n^2 \geq 4 \) is false (the “false implies false” case); similarly, it is true for \( n = 2 \), when \( n \geq 3 \) is false and \( n^2 \geq 4 \) is true (the “false implies true” case). In short, both “false implies true” and “false implies false” are considered true.

We now focus on the mechanics of using the five basic truth tables to compute the truth of compound sentences. This analysis is based on both the truth value of the component sentences and the logical connectives linking them.

**Example 1.2.1** We compute the truth table for \((\neg p) \lor q\).

The two sentence variables \( p \) and \( q \) generate the \( 2 \times 2 = 2^2 = 4 \) permutations of truth values \( TT, TF, FT, FF \) in the corresponding truth table. After listing these permutations, we begin with the innermost connective (the connective farthest inside the parentheses—in this case the negation \( \neg \) on \( p \)) and work our way out through any other connectives (in this case, the disjunction \( \lor \)). We compute one row at a time, applying the corresponding basic truth tables to the particular truth values given in the appropriate columns of the truth table. For this sentence, the operation of the innermost connective (the negation of \( p \) with truth values in the first column) is given in the third column. The effect of the next connective (the disjunction of the third and second columns) follows in the
final (fourth) column.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\sim p$</th>
<th>$(\sim p) \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
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</table>

Example 1.2.2  We compute the truth table for $(\sim p) \land p$.

The one sentence variable $p$ generates the two rows of the corresponding truth table. As in example 1.2.1, the innermost connective is $\sim$ and the outermost is $\land$. First, the operation of the innermost connective (the negation of $p$ with truth values in the first column) is given in the second column. The effect of the next connective (the conjunction of the second and first columns) follows in the final (third) column.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\sim p$</th>
<th>$(\sim p) \land p$</th>
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<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
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Example 1.2.3  We compute the truth table for $(p \land q) \rightarrow r$.

The three distinct statement variables $p$, $q$, and $r$ generate the $2 \times 2 \times 2 = 2^3 = 8$ permutations of truth values in the corresponding truth table. For this sentence, the innermost connective is $\land$ and the outermost is $\rightarrow$. The construction of the truth table proceeds as above, starting with the computation for the innermost connective (the conjunction of the first and second columns) in the fourth column and working outward to the next connective (the implication of the fourth and third columns) in the final (fifth) column.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$p \land q$</th>
<th>$(p \land q) \rightarrow r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
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As can be seen from these three examples, the number of variables in a sentence determines the number of rows in the corresponding truth table. In fact, if a sentence has $n$ variables, the truth table for the sentence has $2^n$ rows. The proof of this numerical relationship uses mathematical induction and is discussed in section 3.6.
Example 1.2.4 Another variation of the truth table question occurs in the context of sentences blending sentence symbols (which have a fixed, known truth value) with sentence variables (which are unspecified and may be either true or false). For example, if $A$ has truth value $T$ and $B$ has truth value $F$, we compute the corresponding truth table for $(A \lor p) \rightarrow B$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$A$</th>
<th>$B$</th>
<th>$A \lor p$</th>
<th>$(A \lor p) \rightarrow B$</th>
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<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
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Question 1.2.1 Compute the truth table for each formal sentence.

(a) $(\sim p) \lor p$  
(b) $(\sim p) \land (\sim q)$

Reflecting on the previous examples and questions, notice that some of the truth tables we have computed possess interesting and important features. In example 1.2.2, we found that the truth table for $(\sim p) \land p$ has all $F$’s in the its final column. Similarly, in question 1.2.1, the truth table for $(\sim p) \lor p$ has all $T$’s in its final column. These are special events for sentences and (as with many special events) such sentences are given distinctive names.

Definition 1.2.1 • A **tautology** is a sentence that has truth value $T$ for every assignment of truth values to its sentence variables.

• A **contradiction** is a sentence that has truth value $F$ for every assignment of truth values to its sentence variables.

• A **contingency** is a sentence that has truth value $T$ for at least one assignment of truth values to its sentence variables and truth value $F$ for at least one assignment of truth values to its sentence variables.

Example 1.2.5 From question 1.2.1, the truth table for $(\sim p) \lor p$ has all $T$’s in its final column, and so $(\sim p) \lor p$ is a tautology. From example 1.2.2, the truth table for $(\sim p) \land p$ has all $F$’s in the its final column, and so $(\sim p) \land p$ is a contradiction. From example 1.2.1, the truth table for $(\sim p) \lor q$ has both $T$’s and $F$’s in its final column, and so $(\sim p) \lor q$ is a contingency.

Question 1.2.2 Compute the truth table for each sentence and identify each as a tautology, a contradiction, or a contingency.

(a) $p \leftrightarrow (\sim p)$  
(b) $p \leftrightarrow p$  
(c) $p \leftrightarrow (p \lor q)$  
(d) $p \leftrightarrow (p \land q)$

We finish this section by defining an important relationship between sentences based on their truth tables. When two sentences have identical final columns in their respective truth tables, we identify them as “the same” in the algebra of logic. This insight motivates the following definition.
Definition 1.2.2  

**Sentences** $B$ and $C$ are logically equivalent if the standard truth tables for $B$ and $C$ have the same final column. We write $B \equiv C$ to denote that $B$ and $C$ are logically equivalent.

The use of the word “if” in mathematical definitions (as in the preceding definition of logical equivalence) is a common practice in mathematical discourse and is always interpreted to mean “if and only if.” This broader interpretation of “if” is used only in the context of definitions, while for theorems, lemmas, and other mathematical statements, we adhere to the strict, formal interpretation of the if–then logical connective. Thus, when we are reading a mathematical definition and encounter the word “if,” we read the definition as an “if and only if” statement asserting the exact meaning of the identified word, allowing us to move freely back and forth between the defined word and the definition.

For example, if two sentences are logically equivalent, then the two sentences have the same final column in their standard truth tables. In addition, if two sentences have the same final column in their standard truth tables, then the two sentences are logically equivalent. You will want to develop a facility in this process of transitioning back and forth between defined mathematical words and the corresponding formal definitions.

We develop a good understanding of logical equivalences by considering some pairs of sentences that are logically equivalent, and some that are not.

**Example 1.2.6**  

We prove that $(p \rightarrow q) \equiv [(\neg p) \lor q]$.

The basic truth table for the implication $p \rightarrow q$ and the standard truth table for $(\neg p) \lor q$ given in example 1.2.1 have the same final columns, as demonstrated below.

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<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
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<tbody>
<tr>
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<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$(\neg p) \lor q$</th>
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<td>$T$</td>
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**Example 1.2.7**  

We prove that both $[(\neg p) \lor p] \not\equiv [(\neg p) \lor q]$ and $[(\neg p) \lor p] \not\equiv (p \rightarrow q)$.

Using the result of example 1.2.6, neither $(p \rightarrow q)$ nor $[(\neg p) \lor q]$ is logically equivalent to a contradiction. A contradiction has truth value $F$ in every row of the final column of its standard truth table, while both of these sentences have $T$ in the first row (and also in the third and fourth rows) of their respective final columns. In example 1.2.2, we found that $(\neg p) \land p$ is a contradiction. Alternatively, observe that the first sentence in each pair has one sentence variable, while the second sentence has two sentence variables, and so they cannot be logically equivalent.

A particularly important pair of logical equivalences is referred to as De Morgan’s laws in honor of the nineteenth century English mathematician Augustus De Morgan, who first identified the significance of these relations for mathematical logic, set theory,
and general mathematical discourse. De Morgan was born in India while his father was serving as an officer in the military, and shortly after birth lost sight in his right eye. While a child, he showed no particular aptitude for academics or athletics, but in 1823 he entered Trinity College of Cambridge University. In 1827, while only 21 years old, De Morgan was appointed as the first professor of mathematics at the newly founded University College London. As a research mathematician, De Morgan is best known for his contribution to mathematical logic, mathematical induction, and the study of algebras. He was also a prolific writer and was a co-founder and the first president of the London Mathematical Society. De Morgan loved mathematical trivia, and noted that he was $x$ years old in the year $x^2$ (he was 43 in 1849); people born in 1980 share this in common with De Morgan (they will be $x = 45$ in $x^2 = 45^2 = 2025$).

**Question 1.2.3 De Morgan’s laws** De Morgan’s laws specify how negation distributes across conjunctions and disjunctions, changing the primary connective. Verify that the sentences in each of the following pairs are logically equivalent by computing the corresponding truth tables.

(a) $[\neg(p \land q)] \equiv [(\neg p) \lor (\neg q)]$

(b) $[\neg(p \lor q)] \equiv [(\neg p) \land (\neg q)]$

### 1.2.1 Reading Questions for Section 1.2

1. State the two truth values of sentential logic. How are they represented?
2. Give an example of a setting in which a three-valued logic might prove useful.
3. State the basic truth tables for the five logical connectives $\neg$, $\land$, $\lor$, $\rightarrow$, and $\leftrightarrow$.
4. Define the standard truth table for a sentence with two variables.
5. What is the relationship between the number of variables in a sentence and the number of rows in the corresponding truth table?
6. Discuss the distinction between an inclusive-or and an exclusive-or.
7. Discuss the definition of the truth table for the implication $p \rightarrow q$.
8. Define and give examples of a tautology, a contradiction, and a contingency.
9. Give natural language examples of a tautology, a contradiction, and a contingency.
10. Define logically equivalent sentences.
11. Give an example of a pair of sentences that are logically equivalent and a pair that are not.
12. State De Morgan’s laws in both sentential logic and English.

### 1.2.2 Exercises for Section 1.2

For exercises 1–20, compute the truth table for each sentence and identify each sentence as a tautology, a contradiction, or a contingency.

1. $p \leftrightarrow (\neg p)$
2. $p \land (p \rightarrow p)$
3. $\neg[(\neg p) \rightarrow p]$
4. $[p \rightarrow (\neg p)] \lor p$
5. $(\neg p) \rightarrow q$
6. $p \leftrightarrow (\neg q)$
In exercises 21–42, determine if each pair of sentences is logically equivalent by computing the corresponding truth tables. Some pairs of sentences have names associated with them to facilitate their use later in the text:

21. Double negation: \( \sim (\sim p); \ p \)
22. De Morgan’s laws: \( \sim (p \land q); \ (\sim p) \lor (\sim q) \)
23. De Morgan’s laws: \( \sim (p \lor q); \ (\sim p) \land (\sim q) \)
24. \( p \land q; \ p \)
25. \( p \lor q; \ p \)
26. Commutativity: \( p \land q; \ q \land p \)
27. Commutativity: \( p \lor q; \ q \lor p \)
28. Associativity: \( (p \land q) \land r; \ p \land (q \land r) \)
29. Associativity: \( (p \lor q) \lor r; \ p \lor (q \lor r) \)
30. \( p \land (q \lor r); \ (p \land q) \lor r \)
31. Distributivity: \( p \land (q \lor r); \ (p \land q) \lor (p \land r) \)
32. Distributivity: \( p \lor (q \land r); \ (p \lor q) \land (p \lor r) \)
33. \( p \lor (q \land r); \ (p \lor q) \land r \)
34. \( (p \to q) \land p; \ q \)
35. Contrapositive: \( p \to q; \ (\sim q) \to (\sim p) \)
36. Inverse: \( p \to q; \ (\sim p) \to (\sim q) \)
37. Converse: \( p \to q; \ q \to p \)
38. Implication expansion: \( p \to q; \ (\sim p) \lor q \)
39. \( p \to q; \ (p \land (\sim q)) \)
40. \( (q \to p); \ (\sim q) \to (\sim p) \)
41. Biconditional expansion: \( p \iff q; \ (p \to q) \land (q \to r) \)
42. \( p \iff q; \ (\sim q) \iff (\sim p) \)

In exercises 43–52, compute the truth table for each sentence under the assumption that sentence symbol \( A \) has truth value \( T \) and sentence symbol \( B \) has truth value \( F \):

43. \( A \to (\sim B) \)
44. \( (A \land B) \lor (\sim B) \)
45. \( A \to p \)
46. \( p \to B \)
47. \( p \to (A \lor B) \)
48. \( p \to (A \land B) \)
49. \( A \iff [p \lor (\sim B)] \)
50. \( (B \land p) \to (\sim A) \)
51. \([\sim (B \land q)] \to (A \iff p) \)
52. \((A \land p) \to (q \lor B) \)

Exercises 53–55 show that logical equivalence is an “equivalence relation” (an important concept discussed in section 2.3) sharing three key properties in common with the standard equality relation =. Verify that = satisfies each property for formal sentences \( B, \ C, \) and \( D \) from sentential logic.

53. Prove \( B \equiv B \).
54. Prove that if \( B \equiv C \), then \( C \equiv B \).
55. Prove that if \( B \equiv C \) and \( C \equiv D \), then \( B \equiv D \).
In exercises 56–57, let $B$ and $C$ be formal sentences from sentential logic and use the definitions of tautology and logical equivalence to prove each statement.

56. If $B \equiv C$, then $B \leftrightarrow C$ is a tautology.
57. If $B \leftrightarrow C$ is a tautology, then $B \equiv C$.

Exercises 58–70 consider the truth functional rendition of the basic truth tables. The basic truth tables can be thought of as defining functions on truth values as illustrated in the following two examples.

$$f_\sim(T) = F \quad f_\sim(F) = T$$
$$f_\land(T, T) = T \quad f_\land(T, F) = F \quad f_\land(F, T) = F \quad f_\land(F, F) = F$$

In exercises 58–60, follow the model given for $f_\sim$ and $f_\land$ and define each truth function on the four distinct ordered pairs of $T$s and $F$s.

58. $f_\lor$
59. $f_\rightarrow$
60. $f_\leftrightarrow$

In exercises 61–66, use the examples and your answers from exercise 58–60, to compute the value of each composite function.

61. $f_\sim(f_\sim(T), F)$
62. $f_\land(f_\sim(T), f_\land(T, T))$
63. $f_\sim(f_\land(T, F), f_\land(F, T))$
64. $f_\sim(f_\sim(T), f_\sim(F))$
65. $f_\sim(f_\land(T), F))$
66. $f_\sim(f_\land(f_\sim(T), F))$

In exercises 67–70, determine the function resulting from each composition or explain why the function is not defined.

67. $f_\sim \circ f_\land$
68. $f_\land \circ f_\sim$
69. $f_\sim \circ f_\lor$
70. $f_\sim \circ f_\lor$

### 1.3 An Algebra for Sentential Logic

In 1854 George Boole published his groundbreaking work *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities* [22]. In this book, Boole developed an algebra of logic for manipulating and simplifying formal sentences. Boole was born in Lincolnshire, England in 1815 and, due to financial constraints, was essentially a self-taught mathematician of extraordinary accomplishments. From the age of 16, Boole supported his parents and siblings by running a series of day and boarding schools. During this time he began studying and researching mathematics, eventually winning the Royal Society’s Royal Medal in 1844 for a paper *On a general method of analysis* applying algebraic methods to solve differential equations. In 1849 Boole was appointed the first professor of mathematics at the newly founded Queen’s College in Cork, Ireland. He taught in Cork for the rest of his life, earning a reputation as an outstanding teacher while remaining a prolific researcher. At the relatively young age of 49, Boole died of a fever after walking from his home to the College in a soaking rainstorm.